Lecture Note-1
Successive Derivatives

Dr. Ganesh C Gorain

J. K. College Purulia

1. Higher order Derivatives :

Let $I := [a, b] \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function defined on I. The derivative of f at $c \in I$, if it exits, is denoted by the symbol $f'(c)$ and is defined as

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}, \quad \text{or equivalently} \quad f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$

If f is a differentiable function on $I_1 \subseteq I$, then f' is the derived function of f from I_1 into $\mathbb{R}$. Again, if f' is a differentiable function on $I_2 \subseteq I_1$, then f'' or $f^{(2)}$ is the derived function of f' from I_2 into $\mathbb{R}$. Similarly, f''' or $f^{(3)}$ is denoted by the derived function of $f^{(2)}$ defined in a suitable interval $I_3 \subseteq I_2$ and so on. In general, $f^{(n)}$ is denoted by the derived function of $f^{(n-1)}$ defined in a suitable interval $I_n \subseteq I_{n-1}$, where $n \in \mathbb{N}$.

In other words, $f^{(n)}(x)$ is the n-th derivative of f with respect to the variable x at a point x in its domain and it is defined as

$$f^{(n)}(x) = \frac{d^n f(x)}{dx^n} = D^n f(x), \quad \forall x \in I_n \subseteq \mathbb{R},\ n \in \mathbb{N} \text{ and } D \equiv \frac{d}{dx}.$$

If $y = f(x)$, then the successive derivatives of y or $f(x)$ with respect to x at a point x in a domain $I_n \subseteq \mathbb{R}$ can be written as

$$\begin{aligned}
y_1 &= Dy = f'(x) = f^{(1)}(x),\\
y_2 &= D^2 y = f''(x) = f^{(2)}(x),\\
y_3 &= D^3 y = f'''(x) = f^{(3)}(x),\\
\cdots & \quad \cdots \quad \cdots\\
y_n &= D^n y = f^n(x) = f^{(n)}(x), \quad n \in \mathbb{N},\ D \equiv \frac{d}{dx}.
\end{aligned}$$

1.1. Examples of Few Standard n-th Derivatives

► **Example 1.** Let $y = f(x) = x^n, \quad n \in \mathbb{N}$.

$$\begin{aligned}
\text{Then} \quad y_1 &= f'(x) = nx^{n-1}\\
y_2 &= f''(x) = n(n-1)x^{n-2}\\
y_3 &= f'''(x) = n(n-1)(n-2)x^{n-3}\\
\cdots &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
y_n &= f^{(n)}(x) = n(n-1)(n-2)\cdots x^{n-n} = n!\\
y_{n+1} &= f^{(n+1)}(x) = 0\\
y_{n+2} &= 0,\ y_{n+3} = 0, \text{ and so on.}
\end{aligned}$$

Thus,

$$D^n(x^n) = \frac{d^n}{dx^n} x^n = n!.$$

Here the result is obtained by inference. But it can be proved by the principle of induction.

▶ **Example 2.** Let $y = f(x) = (ax+b)^\alpha, \quad a, b$ being constants, $\alpha \in \mathbb{R}$.

$$\begin{aligned} \text{Then} \qquad y_1 &= f'(x) = a.\alpha(ax+b)^{\alpha-1} \\ y_2 &= f''(x) = a^2.\alpha(\alpha-1)(ax+b)^{\alpha-2} \\ y_3 &= f'''(x) = a^3.\alpha(\alpha-1)(\alpha-2)(ax+b)^{\alpha-3} \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ y_n &= f^{(n)}(x) = a^n.\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(ax+b)^{\alpha-n}. \end{aligned}$$

Thus,

$$D^n\left((ax+b)^\alpha\right) = \frac{d^n}{dx^n}(ax+b)^\alpha = a^n\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(ax+b)^{\alpha-n}.$$

Problems Set

Establish the followings: $[m, n \in \mathbb{N}, a, b$ being constants, $\alpha \in \mathbb{R}]$

1. $f(x) = (ax+b)^n \Rightarrow f^{(n)}(x) = a^n\, n!$.
2. $f(x) = (ax+b)^m \Rightarrow f^{(n)}(x) = \begin{cases} a^n\, n! \dbinom{m}{n}(ax+b)^{m-n}, & \text{if } n < m; \\ a^m m!, & \text{if } n = m; \\ 0, & \text{if } n > m. \end{cases}$
3. $f(x) = x^{2n} \Rightarrow f^{(n)}(x) = \dfrac{(2n)!}{n!}x^n = n! \dbinom{2n}{n} x^n = 2^n.1.3.5.\cdots.(2n-1).x^n$.
4. $f(x) = \sqrt{x} \Rightarrow f^{(n)}(x) = (-1)^{n-1}\dfrac{1.3.5.\cdots(2n-3)}{2^n x^{n-1/2}}, \ (n > 1)$.
5. $f(x) = 1/\sqrt{x} \Rightarrow f^{(n)}(x) = (-1)^n\dfrac{1.3.5.\cdots(2n-1)}{2^n x^{n+1/2}}$.
6. $f(x) = \dfrac{1}{x+a} \Rightarrow f^{(n)}(x) = (-1)^n\, n! \dfrac{1}{(x+a)^{n+1}}$.
7. $f(x) = \dfrac{1}{(x+a)^2} \Rightarrow f^{(n)}(x) = (-1)^n(n+1)!\dfrac{1}{(x-a)^{n+2}}$.
8. $f(x) = \dfrac{x}{(x+a)^2} \Rightarrow f^{(n)}(x) = (-1)^n\, n! \dfrac{x-an}{(x-a)^{n+2}}$.
9. $f(x) = \dfrac{1}{ax+b} \Rightarrow f^{(n)}(x) = (-1)^n\, n!\, a^n \dfrac{1}{(ax+b)^{n+1}}$.
10. $f(x) = \log(x+a) \Rightarrow f^{(n)}(x) = (-1)^{n-1}(n-1)!\dfrac{1}{(x+a)^n}$.

▶ **Example 3.** If $f(x) = \dfrac{x^3}{x^2-a^2}$, show that $f^{(n)}(x) = (-1)^n\, n! \dfrac{a^2}{2}\left[\dfrac{1}{(x-a)^{n+1}} + \dfrac{1}{(x+a)^{n+1}}\right]$

and hence $\qquad f^{(n)}(0) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ -n!/a^{n-1}, & \text{if } n \text{ is odd;} \end{cases} \quad (n > 1).$

Solution : $f(x) = \dfrac{x^3}{x^2-a^2} = \dfrac{x(x^2-a^2)+a^2x}{x^2-a^2} = x + \dfrac{a^2x}{x^2-a^2} = 1 + \dfrac{a^2}{2}\left(\dfrac{1}{x-a} + \dfrac{1}{x+a}\right)$.

Then
$$f'(x) = 1 + \frac{a^2}{2}\left[(-1)(x-a)^{-2} + (-1)(x+a)^{-2}\right]$$
$$f''(x) = 0 + \frac{a^2}{2}\left[(-1)(-2)(x-a)^{-3} + (-1)(-2)(x+a)^{-3}\right]$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$f^{(n)}(x) = \frac{a^2}{2}\left[(-1)^n n!\,(x-a)^{-(n+1)} + (-1)^n n!\,(x+a)^{-(n+1)}\right]$$
$$= (-1)^n\, n!\frac{a^2}{2}\left[\frac{1}{(x-a)^{n+1}} + \frac{1}{(x+a)^{n+1}}\right], \ (n > 1).$$

Putting $x = 0$, we have $f^{(n)}(0) = \frac{1}{2}(-1)^n\, n! a^2\left[\frac{1}{(-a)^{n+1}} + \frac{1}{a^{n+1}}\right] = \frac{(-1)^n n!}{2a^{n-1}}\left[(-1)^{n+1} + 1\right]$
$$= \begin{cases} 0, & \text{if } n \text{ is even;} \\ -n!/a^{n-1}, & \text{if } n \text{ is odd;} \end{cases} \quad (n > 1).$$

Problems Set

Establish the followings : $[n \in \mathbb{N}, a \neq 0, b \neq 0$ being constants$]$

1. $f(x) = \dfrac{1}{x^2 - a^2} \Rightarrow f^{(n)}(x) = \dfrac{(-1)^n\, n!}{2a}\left[\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(x+a)^{n+1}}\right]$
$$\Rightarrow f^{(n)}(0) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ -n!/a^{n+2}, & \text{if } n \text{ is even.} \end{cases}$$

2. $f(x) = \dfrac{x}{x^2 - a^2} \Rightarrow f^{(n)}(x) = \dfrac{(-1)^n n!}{2}\left[\dfrac{1}{(x-a)^{n+1}} + \dfrac{1}{(x+a)^{n+1}}\right]$
$$\Rightarrow f^{(n)}(0) = \begin{cases} -n!/a^{n+1}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

3. $f(x) = \dfrac{1}{(x-a)(x-b)} \Rightarrow f^{(n)}(x) = \dfrac{(-1)^n\, n!}{a-b}\left[\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(x-b)^{n+1}}\right], \ (a \neq b)$
$$\Rightarrow f^{(n)}(0) = \frac{n!}{b-a}\left(\frac{1}{a^{n-1}} - \frac{1}{b^{n-1}}\right), \ (a \neq b).$$

4. $f(x) = \dfrac{x}{(x-a)(x-b)} \Rightarrow f^{(n)}(x) = \dfrac{(-1)^n\, n!}{a-b}\left[\dfrac{a}{(x-a)^{n+1}} - \dfrac{b}{(x-b)^{n+1}}\right], \ (a \neq b)$
$$\Rightarrow f^{(n)}(0) = \frac{n!}{b-a}\left(\frac{1}{a^n} - \frac{1}{b^n}\right), \ (a \neq b).$$

5. $f(x) = x \log \dfrac{x-1}{x+1} \Rightarrow f^{(n)}(x) = (-1)^n (n-2)!\left[\dfrac{x-n}{(x-1)^n} - \dfrac{x+n}{(x+1)^n}\right], \ (n > 1).$
$$\Rightarrow f^{(n)}(0) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ -\dfrac{2n!}{(n-1)}, & \text{if } n \text{ is even;} \end{cases}, \ (n > 1).$$

6. $f(x) = x \log \dfrac{x+a}{x-a} \Rightarrow f^{(n)}(x) = (-1)^n (n-2)!\left[\dfrac{x+an}{(x+a)^n} - \dfrac{x-an}{(x-a)^n}\right], \ (n > 1).$
$$\Rightarrow f^{(n)}(0) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \dfrac{2n!}{(n-1)a^{n-1}}, & \text{if } n \text{ is even;} \end{cases}, \ (n > 1).$$

▶ **Example 4.** Let $y = f(x) = e^{ax+b}, \quad a, b$ being constants.

$$\text{Then} \quad \begin{aligned} y_1 &= f'(x) = ae^{ax+b} \\ y_2 &= f''(x) = a^2e^{ax+b} \\ y_3 &= f'''(x) = a^3e^{ax+b} \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ y_n &= f^{(n)}(x) = a^ne^{ax+b}. \end{aligned}$$

Thus,

$$D^n(e^{ax+b}) = \frac{d^n}{dx^n}e^{ax+b} = a^ne^{ax+b}.$$

In particular, if $a = 1, b = 0, \quad y = f(x) = e^x$, so that $y_n = f^{(n)}(x) = e^x$.

In particular, if $a = -1, b = 0, \quad y = f(x) = e^{-x}$, so that $y_n = f^{(n)}(x) = (-1)^ne^{-x}$.

▶ **Example 5.** Let $y = f(x) = \sin x$.

$$\text{Then} \quad \begin{aligned} y_1 &= f'(x) = \cos x = \sin\left(\frac{\pi}{2} + x\right) \\ y_2 &= f''(x) = \cos\left(\frac{\pi}{2} + x\right) = \sin\left(2\frac{\pi}{2} + x\right) \\ y_3 &= f'''(x) = \cos\left(2\frac{\pi}{2} + x\right) = \sin\left(3\frac{\pi}{2} + x\right) \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ y_n &= f^{(n)}(x) = \sin\left(n\frac{\pi}{2} + x\right). \end{aligned}$$

Thus,

$$D^n(\sin x) = \frac{d^n}{dx^n}\sin x = \sin\left(n\frac{\pi}{2} + x\right) = \begin{cases} (-1)^{n/2}\sin x, & \text{if } n \text{ is even;} \\ (-1)^{(n-1)/2}\cos x, & \text{if } n \text{ is odd.} \end{cases}$$

▶ **Example 6.** Let $y = f(x) = \cos x$.

$$\text{Then} \quad \begin{aligned} y_1 &= f'(x) = \sin x = \cos\left(\frac{\pi}{2} + x\right) \\ y_2 &= f''(x) = \sin\left(\frac{\pi}{2} + x\right) = \cos\left(2\frac{\pi}{2} + x\right) \\ y_3 &= f'''(x) = \sin\left(2\frac{\pi}{2} + x\right) = \cos\left(3\frac{\pi}{2} + x\right) \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ y_n &= f^{(n)}(x) = \cos\left(n\frac{\pi}{2} + x\right). \end{aligned}$$

Thus,

$$D^n(\cos x) = \frac{d^n}{dx^n}\cos x = \cos\left(n\frac{\pi}{2} + x\right) = \begin{cases} (-1)^{n/2}\cos x, & \text{if } n \text{ is even;} \\ (-1)^{(n+1)/2}\sin x, & \text{if } n \text{ is odd.} \end{cases}$$

Alternatively, we have, $\cos x + i\sin x = e^{ix}, \ (i = \sqrt{-1})$. Hence,

$$\begin{aligned} \frac{d^n}{dx^n}(\cos x + i\sin x) &= \frac{d^n}{dx^n}e^{ix} = i^ne^{ix} = i^n(\cos x + i\sin x) \\ &= \begin{cases} (-1)^{n/2}(\cos x + i\sin x), & \text{if } n \text{ is even;} \\ (-1)^{(n-1)/2}i.(\cos x + i\sin x), & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Equating real and imaginary parts, we thus have when n is even,

$$\frac{d^n}{dx^n}\cos x = (-1)^{n/2}\cos x \qquad \text{and} \qquad \frac{d^n}{dx^n}\sin x = (-1)^{n/2}\sin x,$$

and when n is odd

$$\frac{d^n}{dx^n}\cos x = (-1)^{(n+1)/2}\sin x \qquad \text{and} \qquad \frac{d^n}{dx^n}\sin x = (-1)^{(n-1)/2}\cos x.$$

Problems Set

Establish the followings : $[n \in \mathbb{N}, a, b$ being constants$]$

1. $f(x) = e^{-x/a} \Rightarrow f^{(n)}(x) = \dfrac{(-1)^n}{a^n}e^{-x/a}, \quad (a \neq 0).$
2. $f(x) = \sin(ax+b) \Rightarrow f^{(n)}(x) = a^n \sin\left(n\dfrac{\pi}{2} + ax + b\right).$
3. $f(x) = \cos(ax+b) \Rightarrow f^{(n)}(x) = a^n \cos\left(n\dfrac{\pi}{2} + ax + b\right).$
4. $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2} \Rightarrow f^{(n)}(x) = \dfrac{e^x - (-1)^n e^{-x}}{2} = \begin{cases} \sinh x, & \text{if } n \text{ is even;} \\ \cosh x, & \text{if } n \text{ is odd.} \end{cases}$
5. $f(x) = \cosh x = \dfrac{e^x + e^{-x}}{2} \Rightarrow f^{(n)}(x) = \dfrac{e^x + (-1)^n e^{-x}}{2} = \begin{cases} \sinh x, & \text{if } n \text{ is odd;} \\ \cosh, & \text{if } n \text{ is even.} \end{cases}$
6. $f(x) = \sin nx \Rightarrow f^{(n)}(x) = n^n \sin\left(n\dfrac{\pi}{2} + nx\right) \Rightarrow f^{(n)}(\dfrac{\pi}{2}) = 0, \ \forall n,$

   and $f^{(n)}(0) = \begin{cases} (-1)^{\frac{n-1}{2}} n^n, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$
7. $f(x) = \cos nx \Rightarrow f^{(n)}(x) = n^n \cos\left(n\dfrac{\pi}{2} + nx\right) \Rightarrow f^{(n)}\left(\dfrac{\pi}{2}\right) = (-1)^n n^n, \ \forall n,$

   and $f^{(n)}(0) = \begin{cases} (-1)^{\frac{n}{2}} n^n, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$
8. $f(x) = \sin ax + \cos ax \Rightarrow \left(f^{(n)}(x)\right)^2 = a^{2n}\left(1 + (-1)^n \sin 2ax\right),$

   and $f^{(n)}(0) = \begin{cases} (-1)^{\frac{n-1}{2}} a^n, & \text{if } n \text{ is odd;} \\ (-1)^{\frac{n}{2}} a^n, & \text{if } n \text{ is even.} \end{cases}$

▶ **Example 7.** Let $y = f(x) = e^{ax}\sin bx, \quad a, b \in \mathbb{R}$.

Then

$$\begin{aligned}
y_1 = f'(x) &= e^{ax}(a\sin bx + b\cos bx) \\
&= re^{ax}(\sin bx\cos\theta + b\cos bx\sin\theta), \quad \text{where } a = r\cos\theta, b = r\sin\theta \\
&= re^{ax}\sin(bx+\theta) \\
y_2 = f''(x) &= re^{ax}[a\sin(bx+\theta) + b\cos(bx+\theta)] \\
&= r^2e^{ax}[\sin(bx+\theta)\cos\theta + b\cos(bx+\theta)\sin\theta] \\
&= r^2e^{ax}\sin(bx+2\theta) \\
y_3 = f'''(x) &= r^3e^{ax}\sin(bx+3\theta) \\
\cdots \quad \cdots \quad &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
y_n = f^{(n)}(x) &= r^ne^{ax}\sin(bx+n\theta), \quad \text{where } a = r\cos\theta, b = r\sin\theta \\
&= (a^2+b^2)^{n/2}e^{ax}\sin\left(bx + n\tan^{-1}\frac{b}{a}\right), \text{ since } r = \sqrt{a^2+b^2}, \ \tan\theta = b/a.
\end{aligned}$$

Thus,

$$D^n\left(e^{ax}\sin bx\right)=\frac{d^n}{dx^n}\left(e^{ax}\sin bx\right)=(a^2+b^2)^{n/2}e^{ax}\sin\left(bx+n\tan^{-1}\frac{b}{a}\right).$$

► **Example 8.** Let $y=f(x)=e^{ax}\cos bx,\quad a,b\in\mathbb{R}$.

Then

$$\begin{aligned}
y_1=f'(x)&=e^{ax}(a\cos bx-b\sin bx)\\
&=re^{ax}(\cos bx\cos\theta-b\sin bx\sin\theta),\quad \text{where } a=r\cos\theta, b=r\sin\theta\\
&=re^{ax}\cos(bx+\theta)\\
y_2=f''(x)&=re^{ax}[a\cos(bx+\theta)-b\sin(bx+\theta)]\\
&=r^2e^{ax}[\cos(bx+\theta)\cos\theta-b\sin(bx+\theta)\sin\theta]\\
&=r^2e^{ax}\cos(bx+2\theta)\\
y_3=f'''(x)&=r^3e^{ax}\cos(bx+3\theta)\\
\cdots\quad\cdots&\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
y_n=f^{(n)}(x)&=r^ne^{ax}\cos(bx+n\theta),\quad \text{where } a=r\cos\theta, b=r\sin\theta\\
&=(a^2+b^2)^{n/2}e^{ax}\cos\left(bx+n\tan^{-1}\frac{b}{a}\right),\ \text{since } r=\sqrt{a^2+b^2},\ \tan\theta=b/a.
\end{aligned}$$

Thus,

$$D^n\left(e^{ax}\cos bx\right)=\frac{d^n}{dx^n}\left(e^{ax}\cos bx\right)=(a^2+b^2)^{n/2}e^{ax}\cos\left(bx+n\tan^{-1}\frac{b}{a}\right).$$

Alternatively, we have, $\cos bx+i\sin bx=e^{ibx}$, so that
$e^{ax}(\cos bx+i\sin bx)=e^{ax}e^{ibx}=e^{(a+ib)x}=e^{zx}$ say, where $z=a+ib$.
Thus, we have

$$\frac{d^n}{dx^n}e^{ax}(\cos bx+i\sin bx)=\frac{d^n}{dx^n}e^{zx}=z^ne^{zx}=(a+ib)^ne^{zx}=r^ne^{in\theta}e^{zx},$$

where $a+ib=re^{i\theta}$, so that $r=|a+ib|=\sqrt{a^2+b^2}$ and $\tan\theta=b/a$.
Hence from above, we have

$$\begin{aligned}
\frac{d^n}{dx^n}e^{ax}(\cos bx+i\sin bx)&=r^ne^{in\theta}e^{(a+ib)x}=r^ne^{ax}e^{i(bx+n\theta)}\\
&=r^ne^{ax}[\cos(bx+n\theta)+i\sin(bx+n\theta)]\\
&=(a^2+b^2)^{n/2}e^{ax}\left(\cos\left(bx+n\tan^{-1}\frac{b}{a}\right)+i\sin\left(bx+n\tan^{-1}\frac{b}{a}\right)\right).
\end{aligned}$$

Equating real and imaginary parts, we thus have

$$D^n\left(e^{ax}\cos bx\right)=\frac{d^n}{dx^n}\left(e^{ax}\cos bx\right)=(a^2+b^2)^{n/2}e^{ax}\cos\left(bx+n\tan^{-1}\frac{b}{a}\right).$$

$$D^n\left(e^{ax}\sin bx\right)=\frac{d^n}{dx^n}\left(e^{ax}\sin bx\right)=(a^2+b^2)^{n/2}e^{ax}\sin\left(bx+n\tan^{-1}\frac{b}{a}\right).$$

Problems Set

Establish the followings: [$n\in\mathbb{N},\ a,b,\in\mathbb{R},\ c$ being a constant]

1. $f(x)=e^{ax}\sin(bx+c)\Rightarrow f^{(n)}(x)=(a^2+b^2)^{n/2}e^{ax}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right).$

2. $f(x) = e^{ax}\cos(bx+c) \Rightarrow f^{(n)}(x) = (a^2+b^2)^{n/2}e^{ax}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$.
3. $f(x) = e^{ax}\sin^2\frac{b}{2}x \Rightarrow f^{(n)}(x) = \frac{1}{2}e^{ax}\left(a^n - (a^2+b^2)^{n/2}\cos\left(bx+n\tan^{-1}\frac{b}{a}\right)\right)$.
4. $f(x) = e^{ax}\cos^2\frac{b}{2}x \Rightarrow f^{(n)}(x) = \frac{1}{2}e^{ax}\left(a^n + (a^2+b^2)^{n/2}\cos\left(bx+n\tan^{-1}\frac{b}{a}\right)\right)$.
5. $f(x) = e^{x}(\cos ax+\sin ax) \Rightarrow f^{(n)}(x) = (1+a^2)^{n/2}e^{x}(\sin\psi+\cos\psi),\ \psi = ax+n\tan^{-1}a$.
6. $f(x) = e^{-x}(\cos ax+\sin ax) \Rightarrow f^{(n)}(x) = (1+a^2)^{n/2}e^{x}(\sin\psi+\cos\psi),\ \psi = ax-n\tan^{-1}a$.

1.2. Few Special Forms

▶ **Example 9.** Let $y = f(x) = \dfrac{1}{x^2+a^2}, \quad a \in \mathbb{R}-\{0\}$.

$$\text{Here} \quad y = \frac{1}{2ia}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right).$$

$$\text{Then} \quad y_1 = \frac{1}{2ia}\left((-1)(x-ia)^{-2} - (-1)(x+ia)^{-2}\right)$$

$$y_2 = \frac{1}{2ia}\left((-1)(-2)(x-ia)^{-3} - (-1)(-2)(x+ia)^{-3}\right)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$y_n = f^{(n)}(x) = \frac{(-1)^n n!}{2ia}\left(\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right).$$

Putting $x = r\cos\theta,\ a = r\sin\theta$, we have $x - ia = r(\cos\theta - i\sin\theta) = re^{-i\theta}$, and $x+ia = re^{i\theta}$, so that $(x-ia)^{n+1} = r^{n+1}e^{-i(n+1)\theta}$ and $(x+ia)^{n+1} = r^{n+1}e^{i(n+1)\theta}$.

$$\text{Thus,} \quad y_n = \frac{(-1)^n\, n!}{ar^{n+1}}\left(\frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{2i}\right) = \frac{(-1)^n\, n!}{ar^{n+1}}\sin(n+1)\theta$$

$$= \begin{cases} \dfrac{(-1)^n\, n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta, & \text{since } a = r\sin\theta, \\ \dfrac{(-1)^n\, n!}{a}(a^2+x^2)^{-\frac{n+1}{2}}\sin(n+1)\theta, & \text{since } r^2 = a^2+x^2. \end{cases}$$

Alternatively, Putting $x = a\cot\theta$, so that $\dfrac{dx}{d\theta} = -a\mathrm{cosec}^2\theta$, we have

$$y = \frac{1}{a^2\cot^2\theta - a^2} = \frac{1}{a^2}\sin^2\theta$$

$$\text{Then} \quad y_1 = f'(x) = \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = -\frac{1}{a^3}\sin^2\theta\sin 2\theta$$

$$y_2 = f''(x) = \frac{dy_1}{dx} = \frac{dy_1}{d\theta}\Big/\frac{dx}{d\theta}$$

$$= -\frac{1}{a^3}(2\sin\theta\cos\theta\sin 2\theta + 2\sin^2\theta\cos 2\theta)/(-a\mathrm{cosec}^2\theta)$$

$$= (-1)^2\frac{2!}{a^4}\sin^3\theta\sin 3\theta$$

$$y_3 = f'''(x) = \frac{dy_2}{dx} = \frac{dy_2}{d\theta}\Big/\frac{dx}{d\theta}$$

$$= (-1)^2\frac{2!}{a^4}(3\sin^2\theta\cos\theta\sin 3\theta + 3\sin^3\theta\cos 3\theta)/(-a\mathrm{cosec}^2\theta)$$

$$= (-1)^3\frac{3!}{a^5}\sin^4\theta\sin 4\theta$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$y_n = f^{(n)}(x) = (-1)^n\frac{n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta.$$

Thus,

$$D^n\left(\frac{1}{x^2+a^2}\right)=\frac{d^n}{dx^n}\left(\frac{1}{x^2+a^2}\right)=(-1)^n\frac{n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$$
$$=(-1)^n\frac{n!}{a}(x^2+a^2)^{-\frac{n+1}{2}}\sin(n+1)\theta,\quad \text{where }\cot\theta=x/a.$$

In particular, if $a=1$, we have,

$$D^n\left(\frac{1}{x^2+1}\right)=\frac{d^n}{dx^n}\left(\frac{1}{x^2+1}\right)=(-1)^n n!\sin^{n+1}\theta\sin(n+1)\theta$$
$$=(-1)^n n!(x^2+1)^{-\frac{n+1}{2}}\sin(n+1)\theta,\quad \text{where }\cot\theta=x.$$

Problems Set

Establish the followings : $[n\in\mathbb{N},\ a,b\in\mathbb{R}-\{0\}]$

1. $D^n\left(\dfrac{a}{x^2+a^2}\right)=(-1)^n\dfrac{n!}{a^{n+1}}\sin^{n+1}\theta\sin(n+1)\theta$
$=(-1)^n\, n!(x^2+a^2)^{-\frac{n+1}{2}}\sin(n+1)\theta,\quad$ where $\cot\theta=x/a$.
2. $D^n\left(\dfrac{x}{x^2+a^2}\right)=(-1)^n\dfrac{n!}{a^{n+1}}\sin^{n+1}\theta\cos(n+1)\theta$
$=(-1)^n\, n!(x^2+a^2)^{-\frac{n+1}{2}}\cos(n+1)\theta,\quad$ where $\cot\theta=x/a$.
3. $D^n\left(\tan^{-1}\frac{x}{a}\right)=(-1)^{n-1}\frac{(n-1)!}{a^n}\sin^n\theta\sin n\theta$
$=(-1)^{n-1}(n-1)!\,(x^2+a^2)^{-\frac{n}{2}}\sin n\theta,\quad$ where $\cot\theta=x/a$.
4. $D^n\left(\log(x^2+a^2)\right)=2(-1)^{n-1}\dfrac{(n-1)!}{a^n}\sin^n\theta\cos n\theta$
$=2(-1)^{n-1}(n-1)!\,(x^2+a^2)^{-\frac{n}{2}}\cos n\theta,\quad$ where $\cot\theta=x/a$.
5. $D^n\left(\dfrac{x}{x^2+1}\right)=(-1)^n\, n!\,\sin^{n+1}\theta\cos(n+1)\theta$
$=(-1)^n\, n!(x^2+1)^{-\frac{n+1}{2}}\cos(n+1)\theta,\quad$ where $\cot\theta=x$.
6. $D^n\left(\tan^{-1}\frac{1+x}{1-x}\right)=(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta$
$=(-1)^{n-1}(n-1)!\,(x^2+1)^{-\frac{n}{2}}\sin n\theta,\quad$ where $\cot\theta=x$.
7. $D^n\left(\sin^{-1}\frac{2x}{1+x^2}\right)=D^n\left(\cos^{-1}\frac{1-x^2}{1+x^2}\right)=D^n\left(\tan^{-1}\frac{2x}{1-x^2}\right)=2(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta$
$=2(-1)^{n-1}(n-1)!\,(x^2+1)^{-\frac{n}{2}}\sin n\theta,\quad$ where $\cot\theta=x$.
8. $D^n\left(\log(x^2+1)\right)=2(-1)^{n-1}(n-1)!\,\sin^n\theta\cos n\theta$
$=2(-1)^{n-1}(n-1)!\,(x^2+1)^{-\frac{n}{2}}\cos n\theta,\quad$ where $\cot\theta=x$.

► **Example 10.** Let $y=f(x)=\dfrac{1}{x^2-a^2},\quad a\in\mathbb{R}-\{0\}$.

$$\text{Here} \quad y = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right).$$

$$\text{Then} \quad y_1 = \frac{1}{2a}\left((-1)(x-a)^{-2} - (-1)(x+a)^{-2}\right)$$

$$y_2 = \frac{1}{2a}\left((-1)(-2)(x-a)^{-3} - (-1)(-2)(x+a)^{-3}\right)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$y_n = f^{(n)}(x) = \frac{1}{2a}\left((-1)^n n!\,(x-a)^{-(n+1)} - (-1)^n n!\,(x+a)^{-(n+1)}\right)$$

$$= \frac{(-1)^n n!}{2a}\left(\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right).$$

Putting $x = r\cosh\phi,\ a = r\sinh\phi$, we have $x - a = r(\cosh\phi - \sinh\phi) = r\left(\frac{e^{\phi}+e^{-\phi}}{2} - \frac{e^{\phi}-e^{-\phi}}{2}\right)$ $= re^{-\phi}$, and $x + a = re^{\phi}$, so that $(x-a)^{n+1} = r^{n+1}e^{-(n+1)\phi}$ and $(x+a)^{n+1} = r^{n+1}e^{(n+1)\phi}$.

$$\text{Thus,} \quad y_n = \frac{(-1)^n\, n!}{ar^{n+1}}\left(\frac{e^{(n+1)\phi} - e^{-(n+1)\phi}}{2}\right) = \frac{(-1)^n\, n!}{ar^{n+1}}\sinh(n+1)\phi$$

$$= \begin{cases} \dfrac{(-1)^n\, n!}{a^{n+2}}\sinh^{n+1}\phi\sinh(n+1)\phi, & \text{since } a = r\sinh\phi, \\ \dfrac{(-1)^n\, n!}{a}(a^2+x^2)^{-\frac{n+1}{2}}\sinh(n+1)\phi, & \text{since } r^2 = a^2 + x^2. \end{cases}$$

Alternatively, Putting $x = a\coth\phi$, so that $\dfrac{dx}{d\phi} = -a\,\text{cosech}^2\phi$, we have

$$y = \frac{1}{a^2\coth^2\phi - a^2} = \frac{1}{a^2}\sinh^2\phi$$

$$\text{Then} \quad y_1 = f'(x) = \frac{dy}{dx} = \frac{dy}{d\phi}\Big/\frac{dx}{d\phi} = -\frac{1}{a^3}\sinh^2\phi\sinh 2\phi$$

$$y_2 = f''(x) = \frac{dy_1}{dx} = \frac{dy_1}{d\phi}\Big/\frac{dx}{d\phi}$$

$$= -\frac{1}{a^3}(2\sinh\phi\cosh\phi\sinh 2\phi + 2\sinh^2\phi\cosh 2\phi)/(-a\,\text{cosech}^2\phi)$$

$$= (-1)^2\frac{2!}{a^4}\sinh^3\phi\sinh 3\phi$$

$$y_3 = f'''(x) = \frac{dy_2}{dx} = \frac{dy_2}{d\phi}\Big/\frac{dx}{d\phi}$$

$$= (-1)^2\frac{2!}{a^4}(3\sinh^2\phi\cosh\phi\sinh 3\phi + 3\sinh^3\phi\cosh 3\phi)/(-a\,\text{cosech}^2\phi)$$

$$= (-1)^3\frac{3!}{a^5}\sinh^4\phi\sinh 4\phi$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$y_n = f^{(n)}(x) = (-1)^n\frac{n!}{a^{n+2}}\sinh^{n+1}\phi\sinh(n+1)\phi.$$

Thus,

$$D^n\left(\frac{1}{x^2-a^2}\right) = \frac{d^n}{dx^n}\left(\frac{1}{x^2-a^2}\right) = \frac{(-1)^n n!}{2a}\left(\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right)$$

$$= \frac{(-1)^n\, n!}{a^{n+2}}\sinh^{n+1}\phi\sinh(n+1)\phi,$$

$$= \frac{(-1)^n\, n!}{a}(x^2-a^2)^{-\frac{n+1}{2}}\sinh(n+1)\phi, \quad \text{where } \coth\phi = x/a.$$

In particular, if $a = 1$, we have,

$$D^n\left(\frac{1}{x^2-1}\right) = \frac{d^n}{dx^n}\left(\frac{1}{x^2-1}\right) = (-1)^n\, n!\sinh^{n+1}\phi\sinh(n+1)\phi,$$
$$= (-1)^n\, n!\,(x^2-1)^{-\frac{n+1}{2}}\sinh(n+1)\phi, \quad \text{where } \coth\phi = x.$$

Problems Set

Establish the followings : $[n \in \mathbb{N},\ a, b \in \mathbb{R} - \{0\}]$

1. $D^n\left(\dfrac{a}{x^2-a^2}\right) = \dfrac{(-1)^n\, n!}{a^{n+1}}\sinh^{n+1}\phi\sinh(n+1)\phi$
$= (-1)^n\, n!(x^2-a^2)^{-\frac{n+1}{2}}\sin(n+1)\phi, \quad \text{where } \coth\phi = x/a.$
2. $D^n\left(\dfrac{x}{x^2-a^2}\right) = \dfrac{(-1)^n\, n!}{a^{n+1}}\sinh^{n+1}\phi\cos(n+1)\phi$
$= (-1)^n\, n!(x^2-a^2)^{-\frac{n+1}{2}}\cos(n+1)\phi, \quad \text{where } \coth\phi = x/a.$
3. $D^n\left(\log\dfrac{x-a}{x+a}\right) = \dfrac{2(-1)^{n-1}(n-1)!}{a^n}\sinh^n\phi\sinh n\phi$
$= 2(-1)^{n-1}(n-1)!\,(x^2-a^2)^{-\frac{n}{2}}\sin n\phi, \quad \text{where } \coth\phi = x/a.$
4. $D^n\left(x\log\dfrac{x-a}{x+a}\right) = \dfrac{2(-1)^n(n-2)!}{a^{n-1}}\sinh^{n-1}(\sinh n\phi\cosh\phi - n\cosh n\phi\sinh\phi)$
$= 2(-1)^n(n-2)!(x^2-a^2)^{-\frac{n-1}{2}}(x\sinh n\phi - na\cosh n\phi), \text{ where } \coth\phi = x/a,\ (n > 1).$
5. $D^n\left(\log(x^2-a^2)\right) = \dfrac{2(-1)^{n-1}(n-1)!}{a^n}\sinh^n\phi\cosh n\phi$
$= 2(-1)^{n-1}(n-1)!\,(x^2-a^2)^{-\frac{n}{2}}\cos n\phi, \quad \text{where } \coth\phi = x/a.$
6. $D^n\left(\dfrac{x}{x^2-1}\right) = (-1)^n\, n!\sinh^{n+1}\phi\cos(n+1)\phi$
$= (-1)^n\, n!(x^2-1)^{-\frac{n+1}{2}}\cos(n+1)\phi, \quad \text{where } \coth\phi = x.$
7. $D^n\left(\dfrac{x^2}{x^2-1}\right) = (-1)^n\, n!\sinh^{n+1}\phi\sinh(n+1)\phi$
$= (-1)^n an!(x^2-1)^{-\frac{n+1}{2}}\sin(n+1)\phi, \quad \text{where } \coth\phi = x.$
8. $D^n\left(\log\dfrac{x-1}{x+1}\right) = 2(-1)^{n-1}(n-1)!\ \sinh^n\phi\sinh n\phi$
$= 2(-1)^{n-1}(n-1)!\,(x^2-1)^{-\frac{n}{2}}\sin n\phi, \quad \text{where } \coth\phi = x.$

2. Leibnitz's Theorem :

If u and v are two functions, each differentiable n times at a common point in their domain, then the n-th derivative of their product at that point is given by

$$(uv)_n = \sum_{r=0}^{n}\binom{n}{r}u_{n-r}v_r = u_n v + \binom{n}{1}u_{n-1}v_1 + \binom{n}{2}u_{n-2}v_2 + \cdots + \binom{n}{r}u_{n-r}v_r + \cdots + uv_n,$$

where suffixes denote the order of derivative at the common point of the functions $u,\ v,$ and $u_0 = u,\ v_0 = v$.

Proof : The theorem will be proved by mathematical induction.

Let us denote the statement $P(n): \quad (uv)_n = \sum_{r=0}^{n} \binom{n}{r} u_{n-r} v_r, \quad n \in \mathbb{N}.$

To check the validity of $P(1)$, by actual differentiation $(uv)_1 = u_1 v + u v_1 = \sum_{r=0}^{1} \binom{n}{r} u_{n-r} v_r.$

So the statement $P(1)$ is true. Next, let us assume $P(n)$ is true for $n = m \in \mathbb{N}$.

Then we can write $P(m): \quad (uv)_m = \sum_{r=0}^{m} \binom{m}{r} u_{m-r} v_r.$

Differentiate both sides, we get

$$
\begin{aligned}
(uv)_{m+1} = D\left(\sum_{r=0}^{m} \binom{m}{r} u_{m-r} v_r\right) &= \sum_{r=0}^{m} \binom{m}{r} (u_{m-r+1} v_r + u_{m-r} v_{r+1}) \\
&= \sum_{r=0}^{m} \binom{m}{r} u_{m-r+1} v_r + \sum_{r=0}^{m} \binom{m}{r} u_{m-r} v_{r+1} \\
&= \sum_{r=0}^{m} \binom{m}{r} u_{m-r+1} v_r + \sum_{r=1}^{m+1} \binom{m}{r-1} u_{m-r+1} v_r, \\
&\qquad \text{(replacing the dummy variable } r \text{ as } r-1 \text{ in the second sum)} \\
&= u_{m+1} v + \sum_{r=1}^{m} \left(\binom{m}{r} + \binom{m}{r-1} \right) u_{m-r+1} v_r + u v_{m+1} \\
&= \sum_{r=0}^{m+1} \binom{m+1}{r} u_{m+1-r} v_r, \quad \text{since} \quad \binom{m}{r} + \binom{m}{r-1} = \binom{m+1}{r} \\
&=: P(m+1)
\end{aligned}
$$

Hence by mathematical induction, we have

$$
(uv)_n = \sum_{r=0}^{n} \binom{n}{r} u_{n-r} v_r, \quad \text{i.e.,} \quad D^n(u.v) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r} u \, D^r v, \quad \text{for any } n \in \mathbb{N}.
$$

▶ **Example 11.** Let $y = x^n e^{ax}, \quad a \in \mathbb{R}, \ n \in \mathbb{N}.$

Let $u = e^{ax}$ and $v = x^n$. Then we have

$u_r = D^r\left(e^{ax}\right) = a^r e^{ax}$ and $v_r = D^r\left(x^n\right) = \dfrac{n!}{(n-r)!} x^{n-r}, \quad 0 \le r \le n.$

Thus by Leibnitz's theorem,

$$
\begin{aligned}
y_n &= \sum_{r=0}^{n} \binom{n}{r} u_{n-r} v_r \\
&= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} e^{ax} . \frac{n!}{(n-r)!} x^{n-r} \\
&= n! \, e^{ax} \sum_{r=0}^{n} \binom{n}{r} \frac{(ax)^{n-r}}{(n-r)!}.
\end{aligned}
$$

▶ **Example 12.** Let $y = e^{ax} \cos bx, \quad a, b \in \mathbb{R}.$

Let $u = e^{ax}$ and $v = \cos bx$. Then we have

$u_r = D^r\left(e^{ax}\right) = a^r e^{ax}$ and $v_r = D^r(\cos bx) = b^r \cos\left(r\dfrac{\pi}{2} + bx\right), \quad 0 \le r \le n.$

Thus by Leibnitz's theorem,

$$\begin{aligned} y_n &= \sum_{r=0}^{n} \binom{n}{r} u_{n-r} v_r \\ &= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} e^{ax} . b^r \cos\left(r\frac{\pi}{2} + bx\right) \\ &= e^{ax} \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \cos\left(r\frac{\pi}{2} + bx\right). \end{aligned}$$

▶ **Example 13.** Let $y = x^n \sin nx, \quad n \in \mathbb{N}$.

Let $u = \sin nx$ and $v = x^n$. Then we have

$u_r = D^r(\sin nx) = n^r \sin\left(r\frac{\pi}{2} + nx\right)$ and $v_r = D^r(x^n) = r!\binom{n}{r} x^{n-r}, \quad 0 \le r \le n.$

Thus by Leibnitz's theorem,

$$\begin{aligned} y_n &= \sum_{r=0}^{n} \binom{n}{r} u_{n-r} v_r \\ &= \sum_{r=0}^{n} \binom{n}{r} n^{n-r} \sin\left((n-r)\frac{\pi}{2} + nx\right) . r! \binom{n}{r} x^{n-r} \\ &= \sum_{r=0}^{n} \binom{n}{r}^2 r! \, (nx)^{n-r} \sin\left((n-r)\frac{\pi}{2} + nx\right). \end{aligned}$$

▶ **Example 14.** Let $y = x \log x$.

Let $u = \log x$ and $v = x$. Then we have

$u_r = D^r(\log x) = \dfrac{(-1)^{r-1}(r-1)!}{x^r}, \;\; 1 \le r \le n,$ and $v_0 = v = x, \; v_1 = 1, \; v_r = 0, \;\; 2 \le r \le n.$

Thus by Leibnitz's theorem,

$$\begin{aligned} y_n = \sum_{r=0}^{n} \binom{n}{r} u_{n-r} v_r &= \sum_{r=0}^{1} \binom{n}{r} u_{n-r} v_r \\ &= \frac{(-1)^{n-1}(n-1)!}{x^n} . x + n \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} . 1 \\ &= \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} \left[-(n-1) + n\right] = \frac{(-1)^n (n-2)!}{x^{n-1}}. \end{aligned}$$

▶ **Example 15.** Let $y = \dfrac{\log x}{x}$.

Let $u = \dfrac{1}{x}$ and $v = \log x$. Then we have

$u_r = D^r\left(\dfrac{1}{x}\right) = \dfrac{(-1)^r r!}{x^{r+1}}, \;\; 0 \le r \le n$

and $v_0 = v = \log x, \; v_r = D^r(\log x) = \dfrac{(-1)^{r-1}(r-1)!}{x^r}, \;\; 1 \le r \le n, \; .$

Thus by Leibnitz's theorem,

$$\begin{aligned}
y_n = \sum_{r=0}^{n}\binom{n}{r}u_{n-r}v_r &= \binom{n}{0}u_n v_0 + \sum_{r=1}^{n}\binom{n}{r}u_{n-r}v_r \\
&= \frac{(-1)^n n!}{x^{n+1}}.\log x + \sum_{r=1}^{n}\binom{n}{r}\frac{(-1)^{n-r}(n-r)!}{x^{n-r+1}}.\frac{(-1)^{r-1}(r-1)!}{x^r} \\
&= \frac{(-1)^n n!}{x^{n+1}}.\log x + \sum_{r=1}^{n}\binom{n}{r}\frac{(-1)^{n-1}(n-r)!r!}{x^{n+1}}.\frac{1}{r} \\
&= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - \sum_{r=1}^{n}\frac{1}{r}\right], \text{ since } \binom{n}{r}(n-r)!r! = n! \\
&= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right].
\end{aligned}$$

▶ **Example 16.** Show that $f^{(2n)}(0) = 0, \quad \forall\, n \in \mathbb{N}\cup 0$, where $f(x) = \tan x$. .

Solution :

$$\begin{aligned}
\text{Let} \qquad & y = \tan x \\
& y_1 = \sec^2 x = (1+\tan^2 x) = (1+y^2) \\
& y_2 = 2yy_1.
\end{aligned}$$

Differentiating $2n$ times by Leibnitz's theorem, we get

$$y_{2n+2} = 2\left[y_{2n+1}y + \binom{2n}{1}y_{2n}y_1 + \binom{2n}{2}y_{2n-1}y_2 + \binom{2n}{3}y_{2n-2}y_3 + \cdots\cdots + y_1 y_{2n}\right].$$

Now, putting $x = 0$, we see that $(y_0)_0 = y(0) = 0$, $(y_2)_0 = 2y(0)y_1(0) = 0$, so that $(y_{2r})_0 = f^{(2r)}(0) = 0, \quad$ for $r = 0, 1$.

Next, we assume that $(y_{2r})_0 = 0, \quad \forall\, 0 \le r \le n$.

Putting $x = 0$ in the above expression, we thus have $(y_{2n+2})_0 = 0$, i.e., $(y_{2r})_0 = 0$ for $r = n+1$.

Hence, by mathematical induction, we obtain $(y_{2n})_0 = f^{(2n)}(0) = 0, \quad \forall\, n \in \mathbb{N}\cup 0$.

▶ **Example 17.** Show that $D^n\left(x^n y^n\right) = n!\sum_{r=0}^{n}\binom{n}{r}^2(-1)^r x^r y^{n-r}$, where $x+y=1$, and $D \equiv \frac{d}{dx}$.

Since $x+y=1$, we have $y = 1-x$.
Let $u = x^n$ and $v = y^n = (1-x)^n$. Then for $0 \le r \le n$, we have
$u_r = D^r x^n = \frac{n!}{(n-r)!}x^{n-r}$ and $v_r = D^r(1-x)^n = (-1)^r\frac{n!}{(n-r)!}(1-x)^{n-r} = (-1)^r\frac{n!}{(n-r)!}y^{n-r}$.
Thus by Leibnitz's theorem,

$$\begin{aligned}
D^n\left(x^n y^n\right) &= \sum_{r=0}^{n}\binom{n}{r}u_{n-r}v_r \\
&= \sum_{r=0}^{n}\binom{n}{r}\frac{n!}{r!}x^r.(-1)^r\frac{n!}{(n-r)!}y^{n-r} \\
&= n!\sum_{r=0}^{n}\binom{n}{r}\frac{n!}{r!\,(n-r)!}(-1)^r x^r y^{n-r}
\end{aligned}$$

$$= n! \sum_{r=0}^{n} \binom{n}{r}^2 (-1)^r x^r y^{n-r}, \quad \text{since } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$
$$= n! \left[y^n - \binom{n}{1}^2 x y^{n-1} + \binom{n}{2}^2 x^2 y^{n-2} - \binom{n}{3}^2 x^3 y^{n-3} + \cdots + (-1)^n x^n \right].$$

Problems Set

Using Leibnitz's theorem, establish the followings : $[m, n \in \mathbb{N},\ a, b \in \mathbb{R},\ D \equiv \frac{d}{dx}]$

1. $D^n \left(\frac{x^n}{1-x} \right) = \frac{n!}{(1-x)^{n+1}}$.
2. $D^n \left(\frac{x^n}{ax+b} \right) = \frac{n!\, b^n}{(ax+b)^{n+1}}$.
3. $D^n \left(x^n (1-x)^m \right) = n!\,(1-x)^m \sum_{r=0}^{n} \binom{n}{r} \binom{m}{r} (-1)^r \left(\frac{x}{1-x} \right)^r \quad m \geq n.$
4. $D^n \left(x^n e^x \right) = e^x \sum_{r=0}^{n} \left(\frac{n!}{r!} \right)^2 \frac{(-1)^r x^r}{(n-r)!} = e^x \sum_{r=0}^{n} \left(\frac{n!}{(n-r)!} \right)^2 \frac{(-1)^{n-r} x^{n-r}}{r!}$.
5. $D^n \left(\frac{x^n}{1+x^2} \right) = n!\, \sin\theta \sum_{r=0}^{n} \binom{n}{r} (-1)^r \cos^r\theta \sin(r+1)\theta$, where $x = \cot\theta$.
6. $D^n \left(\frac{1}{1+x+x^2+x^3} \right) = (-1)^n n!\, \sin^{n+2}\theta \sum_{r=0}^{n} \frac{\sin(r+1)\theta}{(\sin\theta + \cos\theta)^{n-r+1}}$, where $x = \cot\theta$.
7. $D^n \left(x^n y^n \right) = n! \sum_{r=0}^{n} \binom{n}{r}^2 (ax)^r y^{n-r}$, where $y = ax+b$.
8. $D^n \left(x^n y^n \right) = y^n \sum_{r=0}^{n} \binom{n}{r}^2 (an)^r (n-r)!\, x^r$, where $y = e^{ax+b}$.

▶ **Example 18.** Differentiating the identity : $x^m.x^n = x^{m+n},\ m, n \in \mathbb{N},\ m \geq n,$ show that
$$\sum_{r=0}^{n} \binom{m}{r} . \binom{n}{r} = \frac{(m+n)!}{m!\,n!}.$$

Solution : Here $x^n.x^m = x^{m+n}$. Let $u = x^n$ and $v = x^m$. Then we have
$u_r = D^r(x^n) = \frac{n!}{(n-r)!} x^{n-r}$ and $v_r = D^r(x^m) = \frac{m!}{(m-r)!} x^{m-r}$, for $0 \leq r \leq n$.
Thus by Leibnitz's theorem, differentiating n times

$$D^n(x^n.x^m) = \sum_{r=0}^{n} \binom{n}{r} u_{n-r} v_r = D^n\left(x^{m+n}\right)$$
$$\text{or,} \quad \sum_{r=0}^{n} \binom{n}{r} \frac{n!}{r!} x^r . \frac{m!}{(m-r)!} x^{m-r} = \frac{(m+n)!}{m!} x^m$$
$$\text{or,} \quad n! x^m \sum_{r=0}^{n} \binom{n}{r} \binom{m}{r} = \frac{(m+n)!}{m!} x^m \Rightarrow \sum_{r=0}^{n} \binom{m}{r} . \binom{n}{r} = \frac{(m+n)!}{m!\,n!}.$$

Problems Set 8

Establish the followings by differentiating the identities : $[n \in \mathbb{N},\ a, b \in \mathbb{R}]$

1. $x^{2n} = x^n.x^n \Rightarrow \sum_{r=0}^{n} \binom{n}{r}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \frac{(2n)!}{(n!)^2}.$
2. $e^{(a+b)x} = e^{ax}.e^{bx} \Rightarrow \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r = (a+b)^n.$
3. $\sin 2x = 2\sin x.\cos x \Rightarrow \sum_{k=0}^{[n/2]} \binom{n}{2k} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{n-1} = 2^{n-1}$, if n is odd.
4. $\frac{1}{x^2 - a^2} = \frac{1}{x+a}.\frac{1}{x-a} \Rightarrow \sum_{r=0}^{n} \frac{1}{(x-a)^r (x+a)^{n-r}} = \frac{1}{2a}\left[\frac{x+a}{(x-a)^n} - \frac{x-a}{(x+a)^n}\right], \quad a \neq 0.$
5. $\frac{1}{1+x+x^2+x^3} = \frac{1}{1+x}.\frac{1}{1+x^2}$

   $\Rightarrow 2\sin\theta \sum_{r=0}^{n} \sin(r+1)\theta(\sin\theta + \cos\theta)^r = 1 + [\sin(n+1)\theta - \cos(n+1)\theta]\,(\sin\theta + \cos\theta)^{n+1}.$

▶ **Example 19.** In view of the identity : $\tan x = \frac{\sin x}{\cos x}$, show that

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} (-1)^k f^{(n-2k)}(0) = \sin\left(n\frac{\pi}{2}\right), \text{ where } f(x) = \tan x, \text{ and hence}$$

$$f^{(n)}(0) - \binom{n}{2} f^{(n-2)}(0) + \binom{n}{4} f^{(n-4)}(0) - \cdots + (-1)^{\frac{(n-1)}{2}} n f'(0) = (-1)^{\frac{(n-1)}{2}}$$

for any odd positive integer n.

Solution : Here $f(x) = \frac{\sin x}{\cos x}$, so that $f(x)\cos x = \sin x$.
Let $u = f(x)$ and $v = \cos x$. Then we have
$u_r = D^r\,(f(x)) = f^{(r)}(x)$ and $v_r = D^r \cos x = \cos\left(r\frac{\pi}{2} + x\right)$, for $0 \leq r \leq n$.
Thus by Leibnitz's theorem, differentiating n times

$$D^n\,(f(x)\cos x) = \sum_{r=0}^{n} \binom{n}{r} u_{n-r} v_r = D^n(\sin x)$$

$$\text{or,} \quad \sum_{r=0}^{n} \binom{n}{r} f^{(n-r)}(x).\cos\left(r\frac{\pi}{2} + x\right) = \sin\left(n\frac{\pi}{2} + x\right)$$

Putting $x = 0$ both sides, we have

$$\sum_{r=0}^{n} \binom{n}{r} f^{(n-r)}(0) \cos\left(r\frac{\pi}{2}\right) = \sin\left(n\frac{\pi}{2}\right).$$

Since, $\cos\left(r\frac{\pi}{2}\right) = \begin{cases} 0, & \text{if } r \text{ is odd;} \\ (-1)^{r/2}, & \text{if } r \text{ is even,} \end{cases}$
putting $r = 2k$, we have

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} f^{(n-2k)}(0)\cos(k\pi) = \sin\left(n\frac{\pi}{2}\right), \quad \text{where } \left[\frac{n}{2}\right] = \begin{cases} (n-1)/2, & \text{if } n \text{ is odd;} \\ n/2, & \text{if } n \text{ is even,} \end{cases}$$

$$\text{or, } \sum_{k=0}^{[n/2]} \binom{n}{2k} (-1)^k f^{(n-2k)}(0) = \sin\left(n\frac{\pi}{2}\right)$$

$$\text{or, } f^{(n)}(0) - \binom{n}{2} f^{(n-2)}(0) + \binom{n}{4} f^{(n-4)}(0) - \cdots$$

$$\cdots + \left\{ \binom{n}{n-1} (-1)^{(n-1)/2} f'(0), \text{ or, } (-1)^{n/2} f(0) \right\} = \sin\left(n\frac{\pi}{2}\right).$$

Since, $f^{(n)}(0) = 0$, if n is even, and $\sin\left(n\frac{\pi}{2}\right) = \begin{cases} (-1)^{(n-1)/2}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even,} \end{cases}$

therefore for any odd positive integer n, we obtain the result

$$f^{(n)}(0) - \binom{n}{2} f^{(n-2)}(0) + \binom{n}{4} f^{(n-4)}(0) - \cdots + (-1)^{\frac{(n-1)}{2}} n f'(0) = \sin\left(n\frac{\pi}{2}\right) = (-1)^{\frac{(n-1)}{2}}.$$

Problems Set 9

Establish the followings in view of the identities : $[n \in \mathbb{N},\ a \in \mathbb{R}]$

1. $\sec x = \dfrac{1}{\cos x} \Rightarrow \displaystyle\sum_{k=0}^{[n/2]} \binom{n}{2k} (-1)^k f^{(n-2k)}(0) = 0$, where $f(x) = \sec x$.
2. $\operatorname{cosec} x = \dfrac{1}{\sin x} \Rightarrow \displaystyle\sum_{k=0}^{[n/2]} \binom{n}{2k} (-1)^k f^{(n-2k)}(\pi/2) = 0$, where $f(x) = \operatorname{cosec} x$.
3. $\cot x = \dfrac{\cos x}{\sin x} \Rightarrow \displaystyle\sum_{k=0}^{[n/2]} \binom{n}{2k} (-1)^k f^{(n-2k)}(\pi/2) = \cos(n+1)\frac{\pi}{2}$, where $f(x) = \cot x$
   $\Rightarrow f^{(n)}(\pi/2) - \binom{n}{2} f^{(n-2)}(\pi/2) + \binom{n}{4} f^{(n-4)}(\pi/2) - \cdots + (-1)^{\frac{(n-1)}{2}} n f'(\pi/2) = (-1)^{\frac{(n+1)}{2}}$,
   if n is any odd positive integer.
4. $x^n e^{-ax} = \dfrac{x^n}{e^{ax}} \Rightarrow \displaystyle\sum_{r=0}^{n} \binom{n}{r} a^{n-r} f^{(r)}(x) = n!\, e^{-ax}$, where $f(x) = x^n e^{-ax}$.
5. $x^n e^{ax} = \dfrac{x^n}{e^{-ax}} \Rightarrow \displaystyle\sum_{r=0}^{n} (-1)^r \binom{n}{r} a^r f^{(n-r)}(x) = n!\, e^{ax}$, where $f(x) = x^n e^{ax}$.

▶ **Example 20.** If $y = \sqrt{\dfrac{1+x}{1-x}}$, show that $(1-x^2)y_{n+1} - (2nx+1)y_n - n(n-1)y_{n-1} = 0$

Solution : Here $y = \sqrt{\dfrac{1+x}{1-x}}$. Taking log both sides, we have

$$\log y = \frac{1}{2}\left[\log(1+x) - \log(1-x)\right].$$

Differentiating, we have

$$\frac{y_1}{y} = \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right] = \frac{1}{1-x^2}$$
$$\text{or,} \qquad (1-x^2)y_1 = y.$$

Differentiating n times by Leibnitz's theorem, we get

$$y_{n+1}(1-x^2) + \binom{n}{1}y_n.(-2x) + \binom{n}{2}y_{n-1}.(-2) = y_n$$
$$\text{or,} \qquad (1-x^2)y_{n+1} - 2nxy_n - n(n-1)y_{n-1} = y_n$$
$$\text{or,} \qquad (1-x^2)y_{n+1} - (2n+1)xy_n - n(n-1)y_{n-1} = 0.$$

► **Example 21.** If $y^{1/m}+y^{-1/m} = 2x$, show that $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n = 0$

Solution : Here $y^{1/m} + y^{-1/m} = 2x \Rightarrow \left(y^{1/m}\right)^2 - 2xy^{1/m} + 1 = 0 \Rightarrow y^{1/m} = x \pm \sqrt{x^2-1}$. Taking log both sides, we have

$$\log y = m \log\left(x \pm \sqrt{x^2-1}\right).$$

Differentiating, we have

$$\frac{y_1}{y} = \frac{m}{x \pm \sqrt{x^2-1}}.\left[1 \pm \frac{x}{\sqrt{x^2-1}}\right] = \pm\frac{m}{\sqrt{x^2-1}}$$
$$\text{or,} \qquad (x^2-1)y_1^2 = m^2y^2.$$

Differentiating again, we have

$$(x^2-1)2y_1y_2 + 2xy_1^2 = 2m^2yy_1 \Rightarrow (x^2-1)y_2 + xy_1 = m^2y.$$

Differentiating n times by Leibnitz's theorem, we get

$$\left[(x^2-1)y_{n+2} + \binom{n}{1}y_{n+1}.(2x) + \binom{n}{2}y_n.2\right] + \left[xy_{n+1} + \binom{n}{1}y_n.1\right] = m^2y_n$$
$$\text{or,} \qquad [(x^2-1)y_{n+2} + 2nxy_{n+1} + n(n-1)y_{n-2}] + [xy_{n+1} + ny_n] = m^2y_n$$
$$\text{or,} \qquad (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

► **Example 22.** If $I_n(x) = e^{-x}D^n\left(e^x x^n\right)$, show that $xI_n''(x) + (1+x)I_n(x) - nI_n(x) = 0$.

Solution : Let $y = e^x x^n$. Taking log both sides, we have $\log y = 1 + n\log x$. Differentiating with respect to x, we get

$$\frac{y_1}{y} = 1 + \frac{n}{x} \Rightarrow xy_1 = (n+x)y.$$

Differentiating $n+1$ times by Leibnitz's theorem, we get

$$\text{or,} \qquad xy_{n+2} + \binom{n+1}{1}y_{n+1}.1 = (n+x)y_{n+1} + \binom{n+1}{1}y_n.1$$
$$\text{or,} \qquad xy_{n+2} + (n+1)y_{n+1} = (n+x)y_{n+1} + (n+1)y_n$$
$$\text{or,} \qquad xy_{n+2} + (1-x)y_{n+1} - (n+1)y_n = 0$$
$$\text{or,} \qquad xD^2(y_n) + (1-x)D(y_n) - (n+1)y_n = 0.$$

Now, given that $I_n(x) = e^{-x}D^n(e^x x^n) = e^{-x}D^n(y) = e^{-x}y_n \Rightarrow y_n = e^x I_n(x)$. Hence, from above, we have

$$xD^2(e^x I_n(x)) + (1-x)D(e^x I_n(x)) - (n+1)e^x I_n(x) = 0$$

or, $$x[e^x I_n''(x) + 2e^x I_n'(x) + e^x I_n(x)] + (1-x)[e^x I_n'(x) + e^x I_n(x)] - (n+1)e^x I_n(x) = 0$$

or, $$x[I_n''(x) + 2I_n'(x) + I_n(x)] + (1-x)[I_n'(x) + I_n(x)] - (n+1)I_n(x) = 0, \quad \text{since } e^x \neq 0$$

or, $$xI_n''(x) + (x+1)I_n'(x) - nI_n(x) = 0.$$

Problems Set

Establish the followings : $[n \in \mathbb{N},\ a, b, m \in \mathbb{R}\ \ D \equiv \frac{d}{dx}]$

1. $y = e^{mx}x^n \Rightarrow xy_{n+1} - mxy_n - mny_{n-1} = 0.$
2. $y = (x^2-1)^n \Rightarrow (x^2-1)y_{n+1} - n(n+1)y_{n-1} = 0.$
3. $y = x^n e^{1/x} \Rightarrow x^2 y_{n+1} + (nx+1)y_n - ny_{n-1} = 0.$
4. $y = e^{m\tan^{-1}x} \Rightarrow (1+x^2)y_{n+1} + (2nx-m)y_n + n(n-1)y_{n-1} = 0.$
5. $x = \tan(\log y)$, or $\log y = \tan^{-1}x$, or $y = e^{\tan^{-1}x} \Rightarrow (1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0.$
6. $y = a\cos(\log x) + b\sin(\log x) \Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$
7. $\cos^{-1}\left(\frac{y}{a}\right) = \log\left(\frac{x}{n}\right)^n \Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0.$
8. $I_n = e^xD^n\left(e^{-x}x^n\right) \Rightarrow xI_n'' + (1-x)I_n' + nI_n = 0.$

► **Example 23.** If $y = \frac{1}{\sqrt{1+2x}}$, show that $(1+2x)y_{n+1} + (2n+1)y_n = 0$ and $y_n(0) = (-1)^n 1.3.5.\cdots.(2n-1)$.

Solution : Here $y = \frac{1}{\sqrt{1+2x}} \Rightarrow y^2.(1+2x) = 1 \Rightarrow 2yy_1.(1+2x) + y^2.2 = 0 \Rightarrow y_1.(1+2x) + y = 0.$

Differentiating n times by Leibnitz's theorem, we get

$$\left[y_{n+1}(1+2x) + \binom{n}{1}y_n.2\right] + y_n = 0$$

or, $$y_{n+1}(1+2x) + 2ny_n + y_n = 0 \quad \text{or,} \quad (1+2x)y_{n+1} + (2n+1)y_n = 0.$$

Putting $x = 0$, we have from above,

$$y(0) = 1, \quad y_1(0) = -y(0) = -1, \quad y_{n+1}(0) = -(2n+1)y_n(0) \Rightarrow \frac{y_{n+1}(0)}{y_n(0)} = -(2n+1),$$

Putting $n = 1, 2, 3, \cdots, (n-1)$, we get successively

$$\frac{y_2(0)}{y_1(0)} = -3, \quad \frac{y_3(0)}{y_2(0)} = -5, \quad \frac{y_4(0)}{y_3(0)} = -7, \cdots, \frac{y_n(0)}{y_{n-1}(0)} = -(2n-1).$$

Multiplying these, we have

$$\frac{y_n(0)}{y_1(0)} = (-)^{n-1}1.3.5.\cdots.(2n-1) \Rightarrow y_n(0) = (-)^{n-1}1.3.5.\cdots.(2n-1)y_1(0)$$

Thus, $y_n(0) = (-1)^n 1.3.5.\cdots.(2n-1)$, since $y_1(0) = -1$.

▶ **Example 24.** If $y = \tan^{-1} x$, show that $(x^2+1)y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0$ and $y_n(0) = \begin{cases} (-1)^{\frac{n-1}{2}}(n-1)!, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$

Solution : Here $y = \tan^{-1} x \Rightarrow y_1 = \dfrac{1}{x^2+1} \Rightarrow y_1(x^2+1) = 1 \Rightarrow y_2(x^2+1) + 2xy_1 = 0.$
Differentiating n times by Leibnitz's theorem, we get

$$\left[y_{n+2}(x^2+1) + \binom{n}{1}y_{n+1}.(2x) + \binom{n}{2}y_n.2\right] + 2\left[xy_{n+1} + \binom{n}{1}y_n.1\right] = 0$$
$$\text{or,}\quad [y_{n+2}(x^2+1) + 2nxy_{n+1} + n(n-1)y_{n-2}] + 2[xy_{n+1} + ny_n] = 0$$
$$\text{or,}\quad (x^2+1)y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0.$$

Putting $x = 0$, we have from above,

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_{n+2}(0) = -n(n+1)y_n(0) = (n+1).n.(-1)y_n(0),$$

Replacing $n-2$ for n, we get

$$\begin{aligned} y_n(0) &= (n-1).(n-2).(-1)y_{n-2}(0) \\ &= (n-1).(n-2).(n-3).(n-4).(-1)^2 y_{n-4}(0) \\ &= (n-1).(n-2).(n-3).(n-4).(n-5).(n-6).(-1)^3 y_{n-6}(0) \\ &= \cdots \quad \cdots \quad \cdots \\ &= \cdots \quad \cdots \quad \cdots \\ &= (n-1).(n-2).(n-3).(n-4).\cdots\cdots \begin{cases} 2.(-1)^{\frac{n-2}{2}}y_2(0) = 0, & \text{if } n \text{ is even;} \\ 1.(-1)^{\frac{n-1}{2}}y_1(0) = (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \end{cases} \\ &= = \begin{cases} 0, & \text{if } n \text{ is even;} \\ (-1)^{\frac{n-1}{2}}(n-1)!, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

▶ **Example 25.** If $y = (\sin^{-1} x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$ and $y_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 2^{n-1}[(\frac{n-2}{2})!]^2, & \text{if } n \text{ is even.} \end{cases}$

Solution : Here $y = (\sin^{-1} x)^2 \Rightarrow y_1 = 2\sin^{-1} x.\dfrac{1}{\sqrt{1-x^2}} \Rightarrow y_1^2(1-x^2) = 4y$
$\Rightarrow 2y_1y_2(1-x^2) - 2xy_1^2 = 4y_1 \Rightarrow y_2(1-x^2) - xy_1 = 2$
Differentiating n times by Leibnitz's theorem, we get

$$\left[y_{n+2}(1-x^2) + \binom{n}{1}y_{n+1}.(-2x) + \binom{n}{2}y_n.(-2)\right] - \left[xy_{n+1} + \binom{n}{1}y_n.1\right] = 0$$
$$\text{or,}\quad [y_{n+2}(1-x^2) - 2nxy_{n+1} - n(n-1)y_{n-2}] - [xy_{n+1} + ny_n] = 0$$
$$\text{or,}\quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$$

Putting $x = 0$, we have from above,

$$y_1(0) = 0, \quad y_2(0) = 2, \quad y_{n+2}(0) = n^2y_n(0), \quad (n \in \mathbb{N})$$

Replacing $n-2$ for n, we get

$$
\begin{aligned}
y_n(0) &= (n-2)^2 y_{n-2}(0) \\
&= (n-2)^2.(n-4)^2 y_{n-4}(0) \\
&= (n-2)^2.(n-4)^2.(n-6)^2 y_{n-6}(0) \\
&= \cdots \quad \cdots \quad \cdots \\
&= \cdots \quad \cdots \quad \cdots \\
&= (n-2)^2.(n-4)^2.(n-6)^2.\cdots\cdots \begin{cases} 2^2.y_2(0) = 2^2.2, & \text{if } n \text{ is even;} \\ 1^2.y_1(0) = 0, & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} 2.[2.4.6.\cdots.(n-2)]^2 = 2.\left[2^{\frac{n-2}{2}}(\frac{n-2}{2})!\right]^2, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \\
&= \begin{cases} 2^{n-1}[(\frac{n-2}{2})!]^2, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

Problems Set 11

Establish the followings : $[n \in \mathbb{N},\ m \in \mathbb{R}]$

1. $y = \sin^{-1} x \Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$
$\Rightarrow y_n(0) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1^2.3^2.5^2.\cdots.(n-2)^2, & \text{if } n \text{ is odd.} \end{cases}$

2. $y = \left(x + \sqrt{1+x^2}\right)^m \Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0,$
$\Rightarrow y_n(0) = \begin{cases} m(m^2-1^2)(m^2-3^2)(m^2-5^2).\cdots.[m^2-(n-2)^2], & \text{if } n \text{ is odd;} \\ m^2(m^2-2^2)(m^2-4^2)(m^2-6^2).\cdots.[m^2-(n-2)^2], & \text{if } n \text{ is even.} \end{cases}$

3. $y = \cos(m \sin^{-1} x) \Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$
$\Rightarrow y_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{n/2} m^2(m^2-2^2)(m^2-4^2)(m^2-6^2).\cdots.[m^2-(n-2)^2], & \text{if } n \text{ is even.} \end{cases}$

4. $y = \sin(m \sin^{-1} x) \Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0.$
$\Rightarrow y_n(0) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ (-1)^{\frac{n-1}{2}} m(m^2-1^2)(m^2-3^2)(m^2-5^2).\cdots.[m^2-(n-2)^2], & \text{if } n \text{ is odd.} \end{cases}$

5. $y = e^{m \sin^{-1} x} \Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0$
$\Rightarrow y_n(0) = \begin{cases} m(m^2+1^2)(m^2+3^2)(m^2+5^2).\cdots.[m^2+(n-2)^2], & \text{if } n \text{ is odd;} \\ m^2(m^2+2^2)(m^2+4^2)(m^2+6^2).\cdots.[m^2-(n-2)^2], & \text{if } n \text{ is even.} \end{cases}$

6. $y = \cosh(\sin^{-1} x) \Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + 1)y_n = 0$
$\Rightarrow y_n(0) = \begin{cases} (1+2^2)(1+4^2)(1+6^2).\cdots.[1+(n-2)^2], & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$

7. $y = (\cos^{-1} x)^2 \Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$
$\Rightarrow y_n(0) = \begin{cases} -\pi[1.3.5.\cdots.(n-2)]^2, & \text{if } n \text{ is odd;} \\ 2^{n-1}[(\frac{n-2}{2})!]^2, & \text{if } n \text{ is even.} \end{cases}$

8. $y = (\sinh^{-1} x)^2 \Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$
$\Rightarrow y_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{\frac{n-2}{2}} 2[2.4.6.\cdots.(n-2)]^2 = (-1)^{\frac{n-2}{2}} 2^{n-1}[(\frac{n-2}{2})!]^2, & \text{if } n \text{ is even.} \end{cases}$

1.3. Few Special Problems :

▶ **Example 26.** Show that $D^n\left(x^{n-1}\log x\right) = \dfrac{(n-1)!}{x}$.

Solution : Let $y = x^{n-1}\log x \Rightarrow y_1 = (n-1)x^{n-2}\log x + \dfrac{x^{n-1}}{x} \Rightarrow xy_1 = (n-1)y + x^{n-1}$.
Differentiating $(n-1)$ times by Leibnitz's theorem, we get

$$xy_n + \binom{n-1}{1} y_{n-1}.1 = (n-1)y_{n-1} + \frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\right)$$

$$\text{or,} \qquad xy_n + (n-1)y_{n-1} = (n-1)y_{n-1} + (n-1)!, \quad \text{since } \frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\right) = (n-1)!$$

$$\text{or,} \qquad y_n = \frac{(n-1)!}{x}.$$

Alternatively, by mathematical induction: Let us denote the statement

$$P(n) : D^n\left(x^{n-1}\log x\right) = \frac{(n-1)!}{x}, \quad n \in \mathbb{N}.$$

To check the validity of $P(1)$, by actual differentiation $D^1\left(x^0\log x\right) = \dfrac{d}{dx}(\log x) = \dfrac{1}{x}$. So $P(1)$ is true.
Next, let us assume $P(n)$ is true for $n = m \in \mathbb{N}$.
Then we have $D^m\left(x^{m-1}\log x\right) = \dfrac{(m-1)!}{x}$.
Now

$$\begin{aligned} D^{m+1}\left(x^m\log x\right) = D^m\Big(D\left(x^m\log x\right)\Big) &= D^m\left(mx^{m-1}\log x + x^{m-1}\right) \\ &= mD^m\left(x^{m-1}\log x\right) + D^m\left(x^{m-1}\right) \\ &= m.\frac{(m-1)!}{x} + 0 = \frac{m!}{x}. \end{aligned}$$

Therefore the statement $P(n)$ is true for $n = m+1$ also. By mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$.

▶ **Example 27.** Show that $D^n\left(x^n\log x\right) = n!\left[\log x + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots\cdots + \dfrac{1}{n}\right]$.

Solution : Let $y = x^n\log x \Rightarrow y_1 = nx^{n-1}\log x + x^{n-1}$.
Differentiating $n-1$ times, we get

$$y_n = D^{n-1}\left(nx^{n-1}\log x + x^{n-1}\right) \Rightarrow D^n\left(x^n\log x\right) = nD^{n-1}\left(x^{n-1}\log x\right) + D^{n-1}\left(x^{n-1}\right).$$

Denoting $I_n := D^n\left(x^n\log x\right), \ \ n \in \mathbb{N}$, we have from above

$$I_n = nI_{n-1} + (n-1)! \Rightarrow \frac{I_n}{n!} - \frac{I_{n-1}}{(n-1)!} = \frac{1}{n}.$$

Putting $n = 2, 3, \cdots n$, we get successively

$$\begin{aligned} &\frac{I_2}{2!} - \frac{I_1}{1!} = \frac{1}{2} \\ &\frac{I_3}{3!} - \frac{I_2}{2!} = \frac{1}{3} \\ &\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots \\ &\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots \\ &\frac{I_n}{n!} - \frac{I_{n-1}}{(n-1)!} = \frac{1}{n}. \end{aligned}$$

Adding these we obtain

$$\frac{I_n}{n!} - \frac{I_1}{1!} = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \Rightarrow I_n = n! \left[I_1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n} \right].$$

But $I_1 = D^1 \left(x^1 \log x \right) = \log x + 1$. Therefore $D^n \left(x^n \log x \right) = n! \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n} \right]$.

Alternatively, by mathematical induction: Let us denote the statement

$$P(n) : D^n \left(x^n \log x \right) = n! \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n} \right], \quad n \in \mathbb{N}.$$

To check the validity of $P(1)$, by actual differentiation $D^1 \left(x^1 \log x \right) = \frac{d}{dx} (x \log x) = \log x + 1$.
So $P(1)$ is true.
Next, let us assume $P(n)$ is true for $n = m \in \mathbb{N}$.
Then we have $D^m \left(x^m \log x \right) = m! \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{m} \right]$.
Now

$$\begin{aligned} D^{m+1} \left(x^{m+1} \log x \right) = D^m \Big(D \left(x^{m+1} \log x \right) \Big) &= D^m \Big((m+1) x^m \log x + x^m \Big) \\ &= (m+1) D^m \left(x^m \log x \right) + D^m \left(x^m \right) \\ &= (m+1) m! \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{m} \right] + m! \\ &= (m+1)! \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{m} + \frac{1}{m+1} \right]. \end{aligned}$$

Therefore the statement $P(n)$ is true for $n = m+1$ also. By mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$.

▶ **Example 28.** Show that $D^n \left(x^{n-1} e^{1/x} \right) = (-1)^n \frac{e^{1/x}}{x^{n+1}}$.

Solution : Let $y = x^{n-1} e^{1/x} \Rightarrow \log y = (n-1) \log x + \frac{1}{x} \Rightarrow \frac{y_1}{y} = \frac{n-1}{x} - \frac{1}{x^2}$
$\Rightarrow x y_1 = (n-1) y - y/x \Rightarrow x y_1 = (n-1) y - x^{n-2} e^{1/x}$.
Differentiating $n-1$ times by Leibnitz theorem, we get
$x y_n + (n-1) y_{n-1} = (n-1) y_{n-1} - D^{n-1} \left(x^{n-2} e^{1/x} \right) \Rightarrow x D^n \left(x^{n-1} e^{1/x} \right) = -D^{n-1} \left(x^{n-2} e^{1/x} \right)$.
Denoting $I_n := D^n \left(x^{n-1} e^{1/x} \right)$, $n \in \mathbb{N}$, we have from above

$$x I_n = -I_{n-1} \Rightarrow \frac{I_n}{I_{n-1}} = -\frac{1}{x}.$$

Putting $n = 2, 3, \cdots n$, we get successively

$$\frac{I_2}{I_1} = -\frac{1}{x}$$
$$\frac{I_3}{I_2} = -\frac{1}{x}$$
$$\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots$$
$$\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots$$
$$\frac{I_n}{I_{n-1}} = -\frac{1}{x}.$$

Multiplying these we obtain

$$\frac{I_n}{I_1} = (-1)^{n-1}\frac{1}{x^{n-1}} \Rightarrow I_n = (-1)^{n-1}\frac{I_1}{x^{n-1}}.$$

But $I_1 = D^1\left(x^0 e^{1/x}\right) = D\left(e^{1/x}\right) = -\frac{1}{x^2}e^{1/x}$. Therefore $\quad D^n\left(x^{n-1}e^{1/x}\right) = (-1)^n\frac{e^{1/x}}{x^{n+1}}$.

Alternatively, by mathematical induction: Let us denote the statement

$$P(n) : D^n\left(x^{n-1}e^{1/x}\right) = (-1)^n\frac{e^{1/x}}{x^{n+1}}, \quad n \in \mathbb{N}.$$

To check the validity of $P(1)$, by actual differentiation $D^1\left(x^0 e^{1/x}\right) = \frac{d}{dx}\left(e^{1/x}\right) = -\frac{1}{x^2}e^{1/x}$. So $P(1)$ is true.

Next, let us assume $P(n)$ is true for $n = m, m-1 \in \mathbb{N}$.

Then we have $D^m\left(x^{m-1}e^{1/x}\right) = (-1)^m\frac{e^{1/x}}{x^{m+1}}$ and $D^{m-1}\left(x^{m-2}e^{1/x}\right) = (-1)^{m-1}\frac{e^{1/x}}{x^m}$.

Now

$$\begin{aligned}
D^{m+1}\left(x^m e^{1/x}\right) = D^m\Big(D\left(x^m e^{1/x}\right)\Big) &= D^m\Big(mx^{m-1}e^{1/x} - x^{m-2}e^{1/x}\Big)\\
&= mD^m\left(x^{m-1}e^{1/x}\right) - D^m\left(x^{m-2}e^{1/x}\right)\\
&= mD^m\left(x^{m-1}e^{1/x}\right) - D\left(D^{m-1}\left(x^{m-2}e^{1/x}\right)\right)\\
&= m(-1)^m\frac{e^{1/x}}{x^{m+1}} - D\left((-1)^{m-1}\frac{e^{1/x}}{x^m}\right)\\
&= m(-1)^m\frac{e^{1/x}}{x^{m+1}} + (-1)^{m-1}\frac{e^{1/x}}{x^{m+2}} - m(-1)^{m-1}\frac{e^{1/x}}{x^{m+1}}\\
&= (-1)^{m-1}\frac{e^{1/x}}{x^{m+2}} = (-1)^{m+1}\frac{e^{1/x}}{x^{m+2}}.
\end{aligned}$$

Therefore the statement $P(n)$ is true for $n = m+1$ also. By mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$.

▶ **Example 29.** Show that $D^n\left(x^{n-1}e^{-\frac{1}{x}}\right) = \frac{e^{-\frac{1}{x}}}{x^{n+1}}$.

Solution : Same as above.

▶ **Example 30.** Show that $D^n\left(\frac{\log x}{x}\right) = (-1)^n\frac{n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots\cdots - \frac{1}{n}\right]$.

Solution : Let $y = \frac{\log x}{x} \Rightarrow xy = \log x$.

Differentiating n times by Leibnitz theorem, we get

$$\begin{aligned}
xy_n + ny_{n-1} = D^n(\log x) = \frac{(-1)^{n-1}(n-1)!}{x^n} &\Rightarrow (-1)^n\left[\frac{x^{n+1}}{n!}y_n + \frac{x^n}{(n-1)!}y_{n-1}\right] = -\frac{1}{n}\\
&\Rightarrow (-1)^n\left[\frac{x^{n+1}}{n!}D^n\left(\frac{\log x}{x}\right) + \frac{x^n}{(n-1)!}D^{n-1}\left(\frac{\log x}{x}\right)\right] = -\frac{1}{n}.
\end{aligned}$$

Denoting $I_n := \frac{x^{n+1}}{n!}D^n\left(\frac{\log x}{x}\right)$, $n \in \mathbb{N}$, we have from above $\quad (-1)^n\Big[I_n + I_{n-1}\Big] = -\frac{1}{n}$.

Putting $n = 2, 3, \cdots n$, we get successively

$$\begin{aligned}
\Big[I_2+I_1\Big] &= -\frac{1}{2}\\
-\Big[I_3+I_2\Big] &= -\frac{1}{3}\\
\Big[I_4+I_3\Big] &= -\frac{1}{4}\\
-\Big[I_5+I_4\Big] &= -\frac{1}{5}\\
\cdots\quad \cdots \qquad & \cdots \quad \cdots\cdots \quad \cdots\\
\cdots\quad \cdots \qquad & \cdots \quad \cdots\cdots \quad \cdots\\
(-1)^n\Big[I_n+I_{n-1}\Big] &= -\frac{1}{n}.
\end{aligned}$$

Adding these we obtain

$$(-1)^n I_n + I_1 = -\frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{n} \Rightarrow I_n = (-1)^n\left[-I_1 - \frac{1}{2} - \frac{1}{3} - \cdots\cdots - \frac{1}{n}\right].$$

But $I_1 = \frac{x^2}{1!}D^1\left(\frac{\log x}{x}\right) = x^2 D\left(\frac{\log x}{x}\right) = x^2\frac{1-\log x}{x^2} = 1-\log x.$

Therefore $\quad D^n\left(\frac{\log x}{x}\right) = (-1)^n\frac{n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} + \cdots\cdots - \frac{1}{n}\right].$

Alternatively, by mathematical induction: Let us denote the statement

$$P(n) : D^n\left(\frac{\log x}{x}\right) = (-1)^n\frac{n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} + \cdots\cdots - \frac{1}{n}\right], \quad n\in\mathbb{N}.$$

To check the validity of $P(1)$, by actual differentiation $D^1\left(\frac{\log x}{x}\right) = \frac{d}{dx}\left(\frac{\log x}{x}\right) = -\frac{1}{x^2}[\log x - 1]$. So $P(1)$ is true.

Next, let us assume $P(n)$ is true for $n = m \in \mathbb{N}$.

Then we have $D^m\left(\frac{\log x}{x}\right) = (-1)^m\frac{m!}{x^{m+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} + \cdots\cdots - \frac{1}{m}\right].$

Now

$$\begin{aligned}
D^{m+1}\left(\frac{\log x}{x}\right) &= D\left(D^m\left(\frac{\log x}{x}\right)\right) = D\left((-1)^m\frac{m!}{x^{m+1}}\Big[\log x - 1 - \frac{1}{2} - \frac{1}{3} + \cdots\cdots - \frac{1}{m}\Big]\right)\\
&= (-1)^{m+1}\frac{m!(m+1)}{x^{m+2}}\Big[\log x - 1 - \frac{1}{2} - \frac{1}{3} + \cdots\cdots - \frac{1}{m}\Big] + (-1)^m\frac{m!}{x^{m+2}}\\
&= (-1)^{m+1}\frac{(m+1)!}{x^{m+2}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} + \cdots\cdots - \frac{1}{m} - \frac{1}{m+1}\right].
\end{aligned}$$

Therefore the statement $P(n)$ is true for $n = m+1$ also. By mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$.