Lecture Note: Fundamental Properties of $\mathbb{R}$

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The set containing all rational numbers as well as irrational numbers is called the set of **Real numbers** and it is denoted by $\mathbb{R}$. In other words, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$, where $\mathbb{Q}$ is the set of rational numbers. The fundamental properties of $\mathbb{R}$ are in the followings:

1. Algebraic Properties
2. Order Properties
3. Completeness Properties
4. Archimedean Properties
5. Density Properties

1.1 Algebraic Properties of $\mathbb{R}$

On the set $\mathbb{R}$, there are two binary operations denoted by $+$ and $.$, called addition and multiplication respectively satisfying the following properties :

A1 : ***Closer property of addition*** *:* $a + b \in \mathbb{R}, \ \forall \ a, b \in \mathbb{R}$.

A2 : ***Associative property of addition*** *:* $(a + b) + c = a + (b + c), \ \forall \ a, b, c \in \mathbb{R}$.

A3 : ***Existence of additive identity (zero element)*** *:* There exits an element 0 in $\mathbb{R}$, $\ni a + 0 = a, \ \forall \ a \in \mathbb{R}$.

A4 : ***Existence of additive inverse (negative element)*** *:* For each element a in $\mathbb{R}$, there exits an element $-a \in \mathbb{R}$, $\ni a + (-a) = 0$.

A5 : ***Commutative property of addition*** *:* $a + b = b + a, \ \forall \ a, b \in \mathbb{R}$.

M1 : ***Closer property of multiplication*** *:* $a.b \in \mathbb{R}, \ \forall \ a, b \in \mathbb{R}$.

M2 : ***Associative property of multiplication*** *:* $(a.b).c = a.(b.c), \ \forall \ a, b, c \in \mathbb{R}$.

M3 : ***Existence of multiplicative identity (unit element)*** *:* There exits an element $1 \in \mathbb{R}$, $\ni a.1 = a, \ \forall \ a \in \mathbb{R}$.

M4 : ***Existence of multiplicative inverse (reciprocal element)*** *:* For each element $a \neq 0$ in $\mathbb{R}$, there exits an element $1/a$ in $\mathbb{R}$, $\ni a.(1/a) = 1$.

M5 : ***Commutative property of multiplication*** *:* $a.b = b.a, \ \forall \ a, b \in \mathbb{R}$.

D : ***Distributive property of multiplication over addition*** *:* $a.(b+c) = a.b+a.c$, $\forall \ a, b, c \in \mathbb{R}$.

It should be remarked that these eleven algebraic properties **A1–A5, M1–M5** and **D** of the set $\mathbb{R}$ of real numbers are the field axioms of $\mathbb{R}$.

Conclusions 1.1.

1. $a+b=a+c \Rightarrow b=c,\ \forall\ a,b,c\in\mathbb{R}$. *: Cancellation law for addition*
   [Hints : $b=0+b=[(-a)+a]+b=(-a)+(a+b)=(-a)+(a+c)=[(-a)+a]+c=0+c=c$.]
2. If $a\neq 0,\ a.b=a.c \Rightarrow b=c,\ \forall\ a,b,c\in\mathbb{R}$. *: Cancellation law for multiplication*
   [Hints : $b=1.b=\left(\frac{1}{a}.a\right).b=\frac{1}{a}.(a.b)=\frac{1}{a}.(a.c)=\left(\frac{1}{a}.a\right).c=1.c=c$.]
3. $a+b=0 \Rightarrow a=-b$ or $b=-a,\ \forall\ a,b\in\mathbb{R}$.
   [Hints : $a+b=0 \Rightarrow a+b=(-b)+b \Rightarrow a=(-b)=-b$.]
4. $-(-a)=a$ for every $a\in\mathbb{R}$.
   [Hints : $a+(-a)=0 \Rightarrow a=-(-a)$.]
5. $a.0=0$ for every $a\in\mathbb{R}$.
   [Hints : $a.0+a.0=a.(0+0)=a.0=a.0+0 \Rightarrow a.0=0$.]
6. $(-1).a=-a$ for every $a\in\mathbb{R}$.
   [Hints : $a+(-1).a=1.a+(-1).a=[1+(-1)].a=0.a=0 \Rightarrow (-1).a=-a$.]
7. $a.(-b)=(-a).b=-(a.b)$ for every $a,b\in\mathbb{R}$.
   [Hints : $a.b+(-a).b=[a+(-a)].b=0.b=0 \Rightarrow (-a).b=-(a.b)$.]
8. $(-a).(-b)=a.b$ for every $a,b\in\mathbb{R}$.
   [Hints : $(-a).(-b)=-[a.(-b)]=-[-(a.b)]=a.b$.]
9. $a.b=1 \Rightarrow a=\frac{1}{b}$, or $b=\frac{1}{a},\ \forall\ a,b\in\mathbb{R}-\{0\}$. [Hints : $a.b=1 \Rightarrow a.b=\frac{1}{b}.b \Rightarrow a=\frac{1}{b}$.]
10. $1/(1/a)=a$ for every $a\neq 0$ in $\mathbb{R}$.
    [Hints : $a.\frac{1}{a}=1 \Rightarrow a=1/(1/a)$.]
11. If $a,b\in\mathbb{R}$, then $a.b=0 \Rightarrow$ either $a=0$ or $b=0$.
    [Hints : $a\neq 0,$ and $a.b=0=a.0 \Rightarrow b=0$. Also $b\neq 0,$ and $a.b=0=0.b \Rightarrow a=0$.]
12. If $a,b\in\mathbb{R}$, and both $a,b\neq 0, \Rightarrow a.b\neq 0$.
    [Hints : $a,b\neq 0,$ and $a.b=0=a.0 \Rightarrow b=0$, a contradiction.]

1.2 Order Properties of $\mathbb{R}$

The order properties of $\mathbb{R}$ refer to the notions of positivity and inequalities between real numbers.

1.2.1 Positivity Properties

There is a non-empty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of **positive real numbers** that satisfy the following properties:

P1 : *Closer property of addition :* If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$.

P2 : *Closer property of multiplication :* If $a, b \in \mathbb{P}$, then $a.b \in \mathbb{P}$.

P3 : *Law of Trichotomy :* If $c \in \mathbb{R}$, then exactly one of the following holds –

$$c \in \mathbb{P}, \quad c = 0, \quad -c \in \mathbb{P}.$$

For every $a \in \mathbb{P}$, we define an order relation '$>$' (read as 'greater than') by $a > 0$ which means that a is a **positive** (or strictly positive) real number. If $a \in \mathbb{P} \cup \{0\}$, we define an order relation ' $\geq$' (read as 'greater than or equal') by $a \geq 0$ which means that a is a **non-negative** real number.

Similarly, if $-a \in \mathbb{P}$, we define an order relation '$<$' (read as 'less than') by $a < 0$ which means that a is a **negative** (or strictly negative) real number. If $-a \in \mathbb{P} \cup \{0\}$, we define an order relation ' $\leq$' (read as 'less than or equal') by $a \leq 0$ which means that a is **non-negative** real number.

For any two $a, b \in \mathbb{R}$, (i) if $a - b \in \mathbb{P}$, then we write either $a > b$, or $b < a$ ($a > b$ is the same as $b < a$) and (ii) if $a - b \in \mathbb{P} \cup \{0\}$, then we write either $a \geq b$, or $b \leq a$ ($a \geq b$ is the same as $b \leq a$). The notations (i) $a \geq b \Rightarrow a > b$, or $a = b$ and (ii) $a \leq b \Rightarrow a < b$, or $a = b$. Hence, $a \geq b,$ and $a \leq b$ together $\Leftrightarrow a = b$.

Remark 1.1. *The set $\mathbb{P}$ together with an order relation is called the set of positive real numbers and it is denoted by $\mathbb{R}^+$. The set $\mathbb{R}^+ \cup \{0\}$ is known as the set of non-negative real numbers. The set $\mathbb{R} - \mathbb{R}^+ \cup \{0\}$ is the set of negative real numbers which is denoted by $\mathbb{R}^-$ and the set $\mathbb{R}^- \cup \{0\}$ is the set of non-positive real numbers.*

We assume that $\exists$ an order relation ' $>'$, or ' $<'$ that defines the ordering of the elements of $\mathbb{R}$ satisfying the following properties :

O1 : *Trichotomy Property :* If $a, b \in \mathbb{R}$, then exactly one of the following holds –

$$a > b, \quad a = b, \quad b > a.$$

O2 : *Transitivity Property :* If $a > b,$ and $b > c$, then $a > c$ for any $a, b, c \in \mathbb{R}$.

O3 : *Compatibility Property :*

(a) If $a > b$, then $a + c > b + c$ for any $a, b, c \in \mathbb{R}$.

(b) If $a > b,$ and $c > 0$, then $a.c > b.c$ for any $a, b \in \mathbb{R}$.

(b) If $a > b,$ and $c < 0$, then $a.c < b.c$ for any $a, b \in \mathbb{R}$.

These fourteen algebraic properties **A1–A5, M1–M5, D** and **O1–O3** make the set $\mathbb{R}$ of real numbers an ordered field.

Conclusions 1.2.

1. If $a > 0 \Rightarrow -a < 0$, and if $a < 0 \Rightarrow -a > 0$ for every $a \in \mathbb{R}$.
   [Hints : $a > 0 \Rightarrow a + (-a) > 0 + (-a) = -a \Rightarrow 0 > -a \Rightarrow -a < 0$.]
2. If $a > 0,\ b > 0 \Rightarrow a + b > 0$, and if $a < 0, b < 0 \Rightarrow a + b < 0$, for every $a, b \in \mathbb{R}$.
   [Hints : $a > 0 \Rightarrow a + b > 0 + b = b$, and then $a + b > b, b > 0 \Rightarrow a + b > 0$.]
3. If $a > 0, b > 0 \Rightarrow a.b > 0$, and if $a < 0, b < 0 \Rightarrow a.b > 0$, for every $a, b \in \mathbb{R}$.
   [Hints : $a > 0, b > 0 \Rightarrow a.b > 0.b = 0 \Rightarrow a.b > 0$.]
4. If $a > 0,\ b < 0 \Rightarrow a.b < 0$, and if $a < 0,\ b > 0 \Rightarrow a.b < 0$, for every $a, b \in \mathbb{R}$.
   [Hints : $b < 0 \Rightarrow -b > 0$, then $a.(-b) > 0 \Rightarrow -(a.b) > 0 \Rightarrow a.b < 0$.]
5. If $a > b,\ c > d \Rightarrow a + c > b + d$ for every $a, b, c, d \in \mathbb{R}$.
   [Hints : $a > b \Rightarrow a + c > b + c$, and $c > d \Rightarrow b + c > b + d$, then $a + c > b + c, b + c > b + d \Rightarrow a + c > b + d$.]
6. If $a > b,\ c > d, a > 0, c > 0 \Rightarrow a.c > b.d$ for every $a, b, c, d \in \mathbb{R}$.
   [Hints : $a > b$, and $c > 0 \Rightarrow a.c > b.c$, and $c > d$, and $a > 0 \Rightarrow a.c > a.d$, then $a.c > b.c, b.c > b.d \Rightarrow a.c > b.d$.]
7. If $a \in \mathbb{R}$, and if $a \neq 0 \Rightarrow a.a = a^2 > 0$.
   [Hints : $a \neq 0 \Rightarrow$ either $a > 0$ or $a < 0$. $a > 0 \Rightarrow a.a > a.0 = 0 \Rightarrow a^2 > 0$, and $a < 0 \Rightarrow -a > 0 \Rightarrow (-a).(-a) > (-a).0 = 0 \Rightarrow a.a > 0 \Rightarrow a^2 > 0$.]
8. If $a > 0 \Rightarrow \dfrac{1}{a} > 0$, and if $a < 0 \Rightarrow \dfrac{1}{a} < 0$ for every $a \in \mathbb{R}$.
   [Hints : $a > 0 \Rightarrow a \neq 0$, so $\exists\ \dfrac{1}{a}(\neq 0) \in \mathbb{R}, \ni a.\dfrac{1}{a} = 1$. Again, $\dfrac{1}{a} \neq 0 \Rightarrow \left(\dfrac{1}{a}\right)^2 > 0$, so that $a > 0 \Rightarrow a.\left(\dfrac{1}{a}\right)^2 > 0 \Rightarrow \left(a.\dfrac{1}{a}\right).\dfrac{1}{a} > 0 \Rightarrow 1.\dfrac{1}{a} > 0 \Rightarrow \dfrac{1}{a} > 0$.]
9. If $a > 0, a.b > c \Rightarrow b > \dfrac{1}{a}.c$, and if $a < 0, a.b > c \Rightarrow b < \dfrac{1}{a}.c$ for every $a, b, c \in \mathbb{R}$.
   [Hints : $a > 0 \Rightarrow \dfrac{1}{a} > 0$, so $a.b > c \Rightarrow \dfrac{1}{a}.(a.b) > \dfrac{1}{a}.c \Rightarrow b > \dfrac{1}{a}.c$.]
10. If $a.b > 0 \Rightarrow$ either $a > 0, b > 0$, or $a < 0, b < 0$ for every $a, b \in \mathbb{R}$.
    [Hints : $a > 0, a.b > 0 \Rightarrow b > \dfrac{1}{a}.0 = 0$.]
11. If $a.b < 0 \Rightarrow$ either $a > 0, b < 0$, or $a < 0, b > 0$ for every $a, b \in \mathbb{R}$.
    [Hints : $a.b < 0 \Rightarrow -(a.b) > 0 \Rightarrow a.(-b) > 0$ so that $a > 0, a.(-b) > 0 \Rightarrow -b > \dfrac{1}{a}.0 = 0 \Rightarrow b < 0$.]

12. If $b > a > 0 \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$, and if $b < a < 0 \Rightarrow \dfrac{1}{a} < \dfrac{1}{b}$ for every $a, b \in \mathbb{R}$.
[Hints : $a, b > 0 \Rightarrow \dfrac{1}{a}, \dfrac{1}{b} > 0 \Rightarrow \dfrac{1}{b}.\dfrac{1}{a} > 0$ so that $b > a \Rightarrow b.\left(\dfrac{1}{b}.\dfrac{1}{a}\right) > a.\left(\dfrac{1}{b}.\dfrac{1}{a}\right) \Rightarrow \left(b.\dfrac{1}{b}\right).\dfrac{1}{a} > a.\left(\dfrac{1}{a}.\dfrac{1}{b}\right) = \left(a.\dfrac{1}{a}\right).\dfrac{1}{b} \Rightarrow \dfrac{1}{a} > \dfrac{1}{b}$.]

1.2.2 The Modulus or Absolute Value

Let $x \in \mathbb{R}$. The **modulus** or **absolute Value** of x denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Thus the numerical value of $|x|$ is either x, or $-x$, or 0 which is non-negative, and $|x| = 0$, if and only if $x = 0$, that means, $|x| \geq 0$ always. Hence, one can write $|x|$ as

$$|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x \leq 0. \end{cases} \quad \Rightarrow \; |x| = \begin{cases} x \geq -x, & \text{if } x \geq 0; \\ -x \geq x, & \text{if } x \leq 0. \end{cases} \quad \Rightarrow \; |x| = \max\{x, -x\}.$$

Geometrically, $|x|$ that is equivalent to $|x - 0|$ is the **distance** between the points x and the origin 0, both lying in the real number line. If general, $|x-y|$ is the **distance** between the points x and the y on the real number line.

Conclusions 1.3.

1. $|-x| = |x|, \; \forall \, x \in \mathbb{R}$.
[Hints : $|-x| = \max\{-x, -(-x)\} = \max\{-x, x\} = |x|$.]
2. $-|x| \leq x \leq |x|, \; \forall \, x \in \mathbb{R}$.
[Hints : $|x| = \max\{x, -x\} \Rightarrow x < |x|$, and $-x < |x| \Rightarrow x < |x|$, and $x > -|x| \Rightarrow -|x| < x < |x|$.]
3. $|x.y| = |x|.|y|, \; \forall \, x, y \in \mathbb{R}$.
[Hints : $|x|.|y| = \max\{x, -x\}.\max\{y, -y\} = \max\{x.y, x.(-y), (-x).y, (-x).(-y)\} = \max\{x.y, -(x.y), -(x.y), x.y\} = \max\{x.y, -(x.y)\} = |x.y|$.]
4. $|x^2| = |x|^2, , \; \forall \, x \in \mathbb{R}$.
[Hints : Particular case $x = y$ of the above conclusion.]
5. $|x/y| = |x|/|y|, \; \forall \, x, y \in \mathbb{R}$, and $y \neq 0$.
[Hints : Writing $1/y$ for y of the above conclusion $\Rightarrow |x/y| = |x.(1/y)| = |x|.|1/y| = |x|/|y|$.]
6. $|x| \leq h \Leftrightarrow -h \leq x \leq h, \; \forall \, x, h \in \mathbb{R}$, and $h \geq 0$.
[Hints : $|x| = \max\{x, -x\} \leq h \Leftrightarrow x \leq h$, and $-x \leq h \Leftrightarrow x \leq h$, and $x \leq -h \Leftrightarrow -h \leq x \leq h$.]

7. $|x \pm y| \leq |x| + |y|, \ \forall \, x, y \in \mathbb{R}$.
[Hints : Adding $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y| \Rightarrow -(|x| + |y|) \leq x + y \leq (|x| + |y|) \Rightarrow |x + y| \leq |x| + |y|$. Again, $|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$.]

8. $\Big||x| - |y|\Big| \leq |x \pm y|, \ \forall \, x, y \in \mathbb{R}$.
[Hints : $|x| = |(x \pm y) \mp y| \leq |x \pm y| + |y| \Rightarrow |x| - |y| \leq |x \pm y|$. Again, $|y| - |x| \leq |y \pm x| \Rightarrow -(|x| - |y|) \leq |x \pm y| \Rightarrow \Big||x| - |y|\Big| \leq |x \pm y|$.]

9. $\Big||x| - |y|\Big| \leq |x \pm y| \leq |x| + |y|, \ \forall \, x, y \in \mathbb{R}$.
[Hints : Combination of above two conclusions.]

10. $|x - a| < \delta \Leftrightarrow a - \delta < x < a + \delta, \ \forall \, x, a, \delta \in \mathbb{R}$, and $\delta > 0$.
[Hints : Putting $u = x - a \Leftrightarrow |u| < \delta \Leftrightarrow -\delta < u < \delta \Leftrightarrow |u| < \delta \Leftrightarrow -\delta < x - a < \delta \Leftrightarrow a - \delta < x < a + \delta$.]

11. $|x| \geq k \Rightarrow$ either $x \geq k$, or $x \leq -k, \ \forall \, x, k \in \mathbb{R}$, and $k \geq 0$.
[Hints : $|x| = \max\{x, -x\} \geq k \Rightarrow \begin{cases} x, & \text{if } x \geq -x; \\ -x, & \text{if } -x \geq x \end{cases} \Rightarrow$ either $x \geq k$, or $-x \geq k \Rightarrow$ either $x \geq k$, or $x \leq -k$.]

12. $|x - a| > k \Rightarrow$ either $x > a + k$, or $x < a - k, \ \forall \, x, a, k \in \mathbb{R}$, and $k > 0$.
[Hints : Putting $u = x - a \Rightarrow |u| > k \Rightarrow$ either $u > k$, or $u < -k \Rightarrow$ either $x - a > k$, or $x - a < -k \Rightarrow$ either $x > a + k$, or $x < a - k$.]

1.3 Completeness Properties of $\mathbb{R}$

This is one of the most fundamental properties of real number, because the field structure axioms and order properties are not sufficient to make a clear idea of real number system. There is more inner property of $\mathbb{R}$ that to stated as axiom, commonly known as **Least upper bound axiom**, or **Supremum property** of $\mathbb{R}$ and its alternate form is known as **Greatest lower bound axiom** or **Infimum property** of $\mathbb{R}$.

A set $S \subset \mathbb{R}$ is said to be **bounded above,** if $\exists$ a $u \in \mathbb{R}, \ni \ x \leq u, \ \forall \ x \in S$. Such number u is called an **upper bound** of S.

The set is said to be **bounded below,** if $\exists$ a $l \in \mathbb{R}, \ni \ x \geq l, \ \forall \ x \in S$. Such number l is called an **lower bound** of S.

A set is said to be **bounded**, if it is both bounded above and bounded below. That means, $\exists \ u, l \in \mathbb{R}, \ni \ l \leq x \leq u, \ \forall \ x \in S$. A set is said to be **unbounded**, if it is not bounded.

Remark 1.2. *The empty set ϕ is also bounded. Every real number is an upper bound as well as lower bound of ϕ.*

Examples 1.1.

1. Let $S = \mathbb{N}$, the set of natural numbers. Then S is bounded below. The number 1 is a lower bound of S. The set S has no upper bound. So it is not bounded above. That means, the set is unbounded, even it is bounded below.
2. Let $S = \mathbb{Z}$, the set of integers. Then the set S has no upper bound as well as no lower bound. So it is neither bounded above nor bounded below. That means, the set is unbounded.
3. Let $S = \left\{ \frac{1}{n} \ : n \in \mathbb{N} \right\}$. Then S is both bounded below and bounded above. The real number 0 is a lower bound of S and the real number 1 is an upper bound of S. That means the set S is bounded.
4. Let $S = \left\{ \frac{(-1)^n}{n} \ : n \in \mathbb{N} \right\}$. Then S is both bounded below and bounded above. The real number -1 is a lower bound of S and the real number $1/2$ is an upper bound of S. That means the set S is bounded.
5. Let $S = \{a_1, a_2, \cdots, a_n\}$, a finite set, where $a_i \in \mathbb{R}$, $\forall\ i = 1, 2, 3, \cdots, n$. Then the set is both bounded below and bounded above so that the set S is bounded. If $\alpha = \min\{a_1, a_2, \cdots, a_n\}$ and $\beta = \max\{a_1, a_2, \cdots, a_n\}$, then α is a lower bound and β is an upper bound of S.
6. $S = (-\infty, 2)$. Then S bounded above. The number 2 is an upper bound of S. The set S has no lower bound. So it is not bounded below. That means, the set is unbounded, even it is bounded above.
7. $S = [5, \infty)$. Then S is bounded below. The number 5 is a lower bound of S. The set S has no upper bound. So it is not bounded above. That means, the set is unbounded, even it is bounded below.
8. $S = (1, 2)$. The real number 1 is a lower bound of S and the real number 2 is an upper bound of S. That means the set S is bounded.
9. $S = [1, 2]$. The real number 1 is a lower bound of S and the real number 2 is an upper bound of S. That means the set S is bounded.
10. Let $S = \mathbb{R}$, the set of real numbers. Then the set is neither bounded below nor bounded above and hence the set S is unbounded.
11. Let $S = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$. Then the set is both bounded below and bounded above so that the set S is bounded. $\frac{1}{2}$ is a lower bound and 1 is an upper bound of S.
12. Let $S = \left\{ \sin \frac{1}{n} \ : \ n \in \mathbb{N} \right\}$. Then the set is both bounded below and bounded above and hence the set S is bounded. -1 is a lower bound and 1 is an upper bound of S.

Remark 1.3. *If a set has one upper bound, then it has infinitely many upper bounds, because if u be an upper bound of S, then the numbers $u+1, u+2, u+3, \cdots$ are also upper bounds of S. In fact, for every $r \in \mathbb{R}^+$, the number $u+r$ will also be an upper bound of S.*

Similarly, if a set has one lower bound, then it has infinitely many lower bounds, because if l be a lower bound of S, then the numbers $l-1, l-2, l-3, \cdots$ are also lower bounds of S. In fact, for every $r \in \mathbb{R}^+$, the number $l-r$ will also be a lower bound of S.

1.3.1 Supremum and Infimum

Let S be a nonempty subset of $\mathbb{R}$.

(**A**) If S is bounded above, then a number $M \in \mathbb{R}$ is said to be the supremum or least upper bound ($l.u.b.$) of S, if it is satisfies the following conditions:

(i) M is an upper bound of S.

(ii) $M \leq$ all upper bounds of S.

The supremum of the set S is denoted by $\sup S$, or $l.u.b.(S)$. The upper bound M of a non-empty set S in $\mathbb{R}$ is the supremum of S, iff for every $\epsilon > 0$, $\exists$ a $v \in S$, $\ni$ $M - \epsilon < v \leq M$.

(**B**) If S is bounded below, then a number $m \in \mathbb{R}$ is said to be the infimum, or greatest lower bound ($g.l.b.$) of S, if it is satisfies the following conditions:

(i) m is a lower bound of S.

(ii) $m \geq$ all lower bounds of S.

The infimum of the set S is denoted by $\inf S$, or $g.l.b.(S)$. The lower bound m of a non-empty set S in $\mathbb{R}$ is the infimum of S, iff for every $\epsilon > 0$, $\exists$ a $v \in S$, $\ni$ $m \leq v < m + \epsilon$.

Remark 1.4. *If S is nonempty and bounded above, then there must exist one and only one supremum of the set S which may not be a member of the set S. Similarly, jf S is nonempty and bounded below, then there must exist one and only one infimum of the set S that may not be a member of the set S.*

The Supremum Properties of $\mathbb{R}$ (The Least Upper Bound Axiom) : Every nonempty subset of $\mathbb{R}$ that has an upper bound also has its supremum in $\mathbb{R}$.

Below is an equivalent or alternative form of the above axiom which is followed from the fact that the set, consisting of all the additive inverses of the elements of a bounded above set, is bounded below.

The Infimum Properties of $\mathbb{R}$ (The Greast Lower Bound Axiom) : Every nonempty subset of $\mathbb{R}$ that has a lower bound also has its infimum in $\mathbb{R}$.

Thus, the set of real numbers is complete in the sense that every nonempty subset of $\mathbb{R}$ bounded above, then it has its infimum in $\mathbb{R}$. Hence, we say that $\mathbb{R}$ is complete ordered field.

Examples 1.2.

1. Let $S = \mathbb{N}$, the set of natural numbers. Then the number 1 is the infimum of S. The set S is not bounded above, so the set has no supremum.
2. Let $S = \mathbb{Z}$, the set of integers. Then the set is neither bounded above nor bounded below. So there is neither supremum nor infimum of the set S.
3. Let $S = \left\{ \frac{1}{n} \ : n \in \mathbb{N} \right\}$. Then S is both bounded below and bounded above. Then the set S is bounded. The number 0 is the infimum and the number 1 is the supremum of S.
4. Let $S = \left\{ \frac{(-1)^n}{n} \ : n \in \mathbb{N} \right\}$. Then S is both bounded below and bounded above. The number -1 is the infimum and the number 1/2 is the supremum of S.
5. Let $S = \{a_1,\ a_2,\ \cdots,\ a_n\}$, a finite set, $a_i \in \mathbb{R},\ \forall\ i = 1, 2, 3, \cdots, n$. Then the set S is bounded. If $\alpha = \min\{a_1,\ a_2,\ \cdots,\ a_n\}$ and $\beta = \max\{a_1,\ a_2,\ \cdots,\ a_n\}$, then α is the infimum and β is the supremum of S.
6. Let $S = (-\infty, 2)$. Then the set S is bounded above. The number 2 is the supremum of S. The set S is not bounded below, so the set has no infimum.
7. Let $S = [5, \infty)$. Then the set is bounded below. The number 5 is the infimum of S. The set S is not bounded above, so the set has no supremum.
8. Let $S = (1, 2)$. Then S is both bounded below and bounded above. Then the set S is bounded. The number 1 is the infimum and the number 2 is the supremum of S.
9. Let $S = [1, 2]$. Then S is both bounded below and bounded above. Then the set S is bounded. The number 1 is the infimum and the number 2 is the supremum of S.
10. Let $S = \mathbb{R}$, the set of real numbers. Then the set S is unbounded. So there is neither supremum nor infimum of the set S.
11. Let $S = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$. Then the set S is bounded. The number $\frac{1}{2}$ is the infimum and the number 1 is the supremum of S.
12. Let $S = \left\{ \sin \frac{1}{n} \ : \ n \in \mathbb{N} \right\}$. Then the set is bounded. The number -1 the infimum and the number 1 is the supremum of S.

1.4 Archimedean Property of $\mathbb{R}$

This is a very natural property of the set of real numbers which help us to record measurements of real number. Without it, one can not imagine the set $\mathbb{R}$ as an ordered field. Thus, an ordered field with this property is called an Archimedean ordered field. It should be remarked that not every ordered field has this property.

Theorem 1.1 (Archimedean Theorem on $\mathbb{R}$). ***If*** $x, y \in \mathbb{R}$, ***and*** $y > 0$, ***then*** $\exists$ ***an*** $n \in \mathbb{N}$, $\ni ny > x$.

Proof. **Case I** : Let $x \le 0$. As we have $y > 0$, and $n > 0$, $\forall\, n \in \mathbb{N} \Rightarrow ny > 0 \ge x \Rightarrow ny \ge x$.

Case II : Let $x > 0$. In this case, if the assertion is false, then $ny \le x$, $\forall\, n \in \mathbb{N}$. Let us define a set $S = \{ny : n \in \mathbb{N}\}$. Then $S \ne \phi$, as $y \in S$, and x is an upper bound of the set S. Therefore by completeness property, the non-empty set S has the supremum. Let $M = \sup S$. So, $ny \le M$ for every $n \in \mathbb{N}$ and $\exists$ a $k \in \mathbb{N}$, $\ni$ $ky > M - y$, since $y > 0$. This implies that $M < (1 + k)y$ which contradicts that M is the supremum of S, since $1 + k \in \mathbb{N}$ and thus $(1 + k)y \in S$. Therefore our assumption $ny \le x$ is wrong. So, $ny > x$. □

Conclusions 1.4.

1. For every $x \in \mathbb{R}$, then $\exists$ an $n \in \mathbb{N}$, $\ni n > x$.
   [Hints : Put $y = 1 > 0$ in the Archimedean Property $ny > x$.]
2. For every $x \in \mathbb{R}$, and $x > 0$, then $\exists$ an $n \in \mathbb{N}$, $\ni 0 < \frac{1}{n} < x$.
   [Hints : As $x > 0$, taking $y = 1$ in the Archimedean Property $nx > y$, we get $nx > 1 \Rightarrow \frac{1}{n}.nx > \frac{1}{n} \Rightarrow x > \frac{1}{n}$.]
   In other words, for any arbitrary $\epsilon > 0$, by Archimedean Property, $\exists$ a natural number n, $\ni 0 < \frac{1}{n} < \epsilon$, which implies that $\lim_{n\to\infty} \frac{1}{n} = 0$.
3. If $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, then $\inf S = 0$.
   [Hints : As $\frac{1}{n} > 0$, $\forall\, n \in \mathbb{N}$, we have $\inf S \ge 0$. Let $m = \inf S$. If $m > 0$, then by Archimedean Property, $\exists$ a $k \in \mathbb{N}$, $\ni 0 < \frac{1}{k} < m$, which contradicts that m is the infimum of S. Hence $m = 0$.]
4. If $x \in \mathbb{R}$, and $x \ge 0$, then $\exists$ a unique $n \in \mathbb{N}$, $\ni n - 1 \le x < n$.
   [Hints : Since $x \ge 0$, by Archimedean Property, $\exists$ a natural number $m : m > x$. Let us define a set $S = \{k \in \mathbb{N} : k > x\}$. Now, obviously $S \ne \phi$, since $m \in S$, and $S \subseteq \mathbb{N}$. By well ordering property of $\mathbb{N}$, the set S has the least element, say $n \in S \subseteq \mathbb{N}$. Thus $n > x$. Since n is the least element of S, therefore $n - 1 \notin S$. So, $n - 1 \le x$. Since the least element n of S is unique, hence we have a unique $n \in \mathbb{N}$, $\ni n - 1 \le x < n$.]

5. For every $x \in \mathbb{R}$, and $x > 0$, then $\exists$ a unique $n \in \mathbb{N}$, $\ni n-1 < x \leq n$.
[Hints : Since $x > 0$, by Archimedean Property, $\exists$ a natural number $m : m > x$. Let $S := \{k \in \mathbb{N} : k \geq x\}$. Now, obviously $S \neq \phi$, since $m \in S$, and $S \subseteq \mathbb{N}$. By well ordering property of $\mathbb{N}$, the set S has the least element, say $n \in S \subseteq \mathbb{N}$. Thus $n \geq x$. Since n is the least element of S, therefore $n-1 \notin S$. So, $n-1 < x$. Since the least element n of S is unique, hence we have a unique $n \in \mathbb{N}$, $\ni n-1 < x \leq n$.]

   The conclusions 4 and 5 together imply that for any real $x \geq 0$, $\exists$ a non-negative integer called $[x]$ ('$[x]$' denotes the greatest integer not greater that x), $\ni$ either $[x]-1 \leq x < [x]$, or $[x]-1 < x \leq [x]$, or $[x] \leq x < [x]+1$ holds.

6. For every $x \in \mathbb{R}$, $\exists$ an $m \in \mathbb{Z}$, $\ni$, $m-1 < x \leq m$, or $m-1 \leq x < m$.
[Hints : If $x \geq 0$, the results are obvious from the conclusions 4 and 5 with $m \in \mathbb{N} \subset \mathbb{Z}$.
If $x < 0$, then $-x > 0$. By the conclusions 4 nd 5, $\exists$ unique $n \in \mathbb{N}$, $\ni n-1 \leq -x < n$, or $n-1 < -x \leq n \Rightarrow -n+1 \geq -x > -n$, or $-n+1 > -x \geq -n \Rightarrow -n < -x \leq -n+1$, or $-n \leq x < -n+1 \Rightarrow m-1 < x \leq m$, or $m-1 \leq x < m$ where $m = -n+1 \in \mathbb{Z}$.

   Hence, in view of the conclusions 4 and 5, we have for any real $x \in \mathbb{R}$, $\exists$ an integer called $[x]$, $\ni [x]-1 \leq x < [x]$, or $[x]-1 < x \leq [x]$ (or simply, $[x] \leq x < [x]+1$).

7. For every $x \in \mathbb{R}$, $\exists$ an $m \in \mathbb{Z}$, $\ni$, $x < m \leq x+1$.
[Hints : Since $x \in \mathbb{R}$, by Archimedean property of $\mathbb{R}$, $\exists$ an $m \in \mathbb{Z}$, $\ni\ m-1 \leq x < m \Rightarrow m \leq 1+x$, or $x < m \Rightarrow x < m \leq x+1$.]

1.5 Density Properties of $\mathbb{R}$

The set of real numbers consists of the set of rational numbers and the set of irrational numbers. The set of rational numbers is dense in $\mathbb{R}$ in the sense that between any two given real numbers, there is a rational number. In fact, there are infinitely many such rational numbers. The set of irrational numbers is also dense in $\mathbb{R}$ in the sense that between any two given real numbers, there is an irrational number. In fact, there are infinitely many such irrational numbers.

Theorem 1.2. ***If*** $x, y \in \mathbb{R}$, $\ni\ x < y$***, then*** $\exists$ ***a rational number*** r, $\ni\ x < r < y$.

Proof. Since $x < y$. so $y - x > 0$. By Archimedean property of $\mathbb{R}$, $\exists$ an $n \in \mathbb{N}$, $\ni$ $0 < 1/n < y - x$. That means, $nx + 1 < ny$.

Again, $nx \in \mathbb{R}$. By Archimedean property of $\mathbb{R}$, $\exists$ an $m \in \mathbb{Z}$, $\ni\ nx < m \leq nx+1$.

Thus it follows that $nx < m < ny$, or $x < m/n < y$. So, $\exists$ a rational number $r := m/n$, $\ni\ x < r < y$. ☐

Theorem 1.3. ***If*** $x, y \in \mathbb{R}$, $\ni\ x < y$***, then*** $\exists$ ***an irrational number*** z, $\ni\ x < z < y$.

Proof. By the above density property, between two real numbers $\sqrt{2}x, \sqrt{2}y$, $\exists$ a rational number $r \neq 0 \ni \sqrt{2}x < r < \sqrt{2}y \Rightarrow x < r/\sqrt{2} < y$. So, $\exists$ an irrational number $z := r/\sqrt{2}$, $\ni\ x < z < y$. □

Remark 1.5. *It should be remarked that between any two real numbers, there lie an infinite number of rational numbers as well as infinite number of irrational numbers and as a whole number infinite number of real numbers.*

1.6 Countability of sets

1.6.1 Enumerable/Denumerable set

A set $S \subseteq \mathbb{R}$ is said to be enumerable, or denumerable or countably infinite, if $\exists$ a bijection on $\mathbb{N}$ onto S.

If $f : \mathbb{N} \to S$ is the bijective mapping, then the elements of a denumerable set S can be expressed as $f(1), f(2), f(3), \cdots$, or $a_1, a_2, a_3, \cdots$, showing that the elements of S are indexed by the set $\mathbb{N}$.

A set $S \subseteq \mathbb{R}$ is said to be countable, if it is either finite, or denumerable.

A set $S \subseteq \mathbb{R}$ is said to be uncountable, if it is not countable.

Examples 1.3.

1. Let $S = \mathbb{N} = \{1,\ 2,\ 3,\ \cdots\}$. Then S is countable, since the mapping $f : \mathbb{N} \to S$ defined by $f(n) = n,\ \forall\ n \in \mathbb{N}$ is a bijective mapping.

2. Let $S = \{2n : n \in \mathbb{N}\}$. Then S is countable, since the mapping $f : \mathbb{N} \to S$ defined by $f(n) = 2n,\ \forall\ n \in \mathbb{N}$ is a bijective mapping.

3. Let $S = \{(2n-1)^2 : n \in \mathbb{N}\}$. Then S is countable, since the mapping $f : \mathbb{N} \to S$ defined by $f(n) = (2n-1)^2,\ \forall\ n \in \mathbb{N}$ is a bijective mapping.

4. Let $S = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \cdots\right\}$. Then S is countable, since the mapping $f : \mathbb{N} \to S$ defined by $f(n) = 1/n,\ \forall\ n \in \mathbb{N}$ is a bijective mapping.

5. Let $S = \mathbb{Z} = \{0,\ 1,\ -1,\ 2,\ -2,\ \cdots\}$. Then S is countable, since the mapping $f : \mathbb{N} \to S$ defined by $f(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (1-n)/2, & \text{if } n \text{ is odd} \end{cases}$ is a bijective mapping.

Theorem 1.4. ***Every infinite subset of a denumerable set is denumerable.***

Proof. Let S be a denumerable set. So its elements can be indexed as $a_1, a_2, a_3, \cdots$.

Let $T \subseteq S$. So the elements T can be indexed from a set $\mathbb{P} \subseteq \mathbb{N}$. By well ordering principle of $\mathbb{N}$, the set $\mathbb{P}$ contains a smallest element, say p_1. Since $p_1 \in \mathbb{P}$, we have $a_{p_1} \in T$.

By well ordering principle of $\mathbb{N}$, the set $\mathbb{P} - \{p_1\}$ contains a smallest element, say p_2. Since $p_2 \in \mathbb{P}$, we have $a_{p_2} \in T$.

Proceeding in this way, we obtain the elements of T as $a_{p_1}, a_{p_2}, a_{p_3}, \cdots$, where $p_1, p_2, p_3, \cdots \in \mathbb{P}$. If we define a mapping $f : \mathbb{N} \to T$ by $f(n) = a_{p_n}$, then it can be shown easily that f is a bijective. Because,

(i) if $i \neq j \Rightarrow p_i \neq p_j \Rightarrow a_{p_i} \neq a_{p_j} \Rightarrow f(i) \neq f(j) \Rightarrow f$ is injective.

(ii) for any $f(r) \in T \Rightarrow a_{p_r} \in T \Rightarrow \exists$ a $p_r \in \mathbb{P} \Rightarrow \exists$ a $r \in \mathbb{N} \Rightarrow f$ is surjective. □

Remark 1.6. *Every subset of a countable set is countable.*

Theorem 1.5. ***The union of two denumerable sets is denumerable.***

Proof. Let $A = \{a_1, a_2, a_3, \cdots\}$ and $B = \{b_1, b_2, b_3, \cdots\}$ are two denumerable sets.

Case I : Let A and B be disjoint, i. e., $A \cap B = \phi$.

Then $A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \cdots\}$

The set $A \cup B$ is countable, since the mapping $f : \mathbb{N} \to A \cup B$ defined by

$$f(n) = \begin{cases} a_{\frac{n+1}{2}}, & \text{if } n \text{ is even,} \\ b_{\frac{n}{2}}, & \text{if } n \text{ is odd} \end{cases}$$

is a bijective mapping.

Case II : Let A and B are not disjoint, i. e., $A \cap B \neq \phi$.

Let $C = B - A$. Then $A \cap C = A \cap B$.

Now A is denumerable, $C \subset B \Rightarrow C$ is also denumerable and $A \cap C = \phi$. By the Case I, $A \cup C$ is denumerable $\Rightarrow A \cup B$ is denumerable. □

Conclusions 1.5.

1. The union of two countable sets is countable.
2. The union of finite number of denumerable sets is denumerable.
3. The union of denumerable number of denumerable sets is denumerable.
4. The union of countable number of countable sets is countable.
5. The set $\left\{\left(\dfrac{1}{m}, \dfrac{1}{n}\right) : m, n \in \mathbb{N}\right\}$ is countable.

Theorem 1.6. ***The Cartesian product of two denumerable sets is denumerable.***

Proof. Let $A = \{a_1,\ a_2,\ a_3,\ \cdots\}$

and $B = \{b_1,\ b_2,\ b_3,\ \cdots\}$ are two denumerable sets in $\mathbb{R}$.. Then

$$A \times B = \left\{ \begin{array}{l} (a_1, b_1),\ (a_1, b_2),\ (a_1, b_3),\ \cdots, (a_2, b_1),\ (a_2, b_2),\ (a_2, b_3),\ \cdots, \\ \\ (a_3, b_1),\ (a_3, b_2),\ (a_3, b_3), \cdots, \cdots, \cdots, \cdots, \cdots, ; \cdots, \cdots, \\ \\ (a_k, b_1),\ (a_k, b_2),\ (a_k, b_3), \cdots, \cdots, \cdots, \cdots, \cdots, ; \cdots, \cdots, \end{array} \right\}$$

Let $A_k = \{(a_k, b_1),\ (a_k, b_2),\ (a_k, b_3),\ \cdots\}$ for every $k = 1, 2, 3, \cdots$.

Then $A \times B = \displaystyle\bigcup_{k=1}^{\infty} A_k$ and each A_k is denumerable.

Therefore, $A \times B$ is denumerable. $\square$

Remark 1.7. *The Cartesian product of two countable sets is countable.*

Theorem 1.7. ***The set $\mathbb{Q}$ of all rational numbers is countable.***

Proof. Let $\mathbb{Q}^+$ is the set of all positive rational numbers and $\mathbb{Q}^-$ is the set of all negative rational numbers. Then $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$.

Then the elements of $\mathbb{Q}^+$ can be expressed as

$$\begin{array}{lllllll}
\frac{1}{1}, & \frac{2}{1}, & \frac{3}{1}, & \frac{4}{1}, & \frac{5}{1}, & \cdots, & \cdots, \\
\frac{1}{2}, & \frac{2}{2}, & \frac{3}{3}, & \frac{4}{2}, & \frac{5}{2}, & \cdots, & \cdots, \\
\frac{1}{3}, & \frac{2}{3}, & \frac{3}{3}, & \frac{4}{3}, & \frac{5}{3}, & \cdots, & \cdots, \\
\cdots, & \cdots, & \cdots, & \cdots, & \cdots, & & \\
\cdots, & \cdots, & \cdots, & \cdots, & \cdots, & & \\
\frac{1}{k}, & \frac{2}{k}, & \frac{3}{k}, & \frac{4}{k}, & \frac{5}{k}, & \cdots, & \cdots, \\
\cdots, & \cdots, & \cdots, & \cdots, & \cdots & & \\
\cdots, & \cdots, & \cdots, & \cdots, & \cdots, & &
\end{array}$$

Let $A_k = \left\{\frac{1}{k},\ \frac{2}{k},\ \frac{3}{k},\ \frac{4}{k},\ \frac{5}{k},\ \cdots\cdots\right\}$ for $k = 1, 2, 3, \cdots$.

Then $\mathbb{Q}^+ = \displaystyle\bigcup_{k=1}^{\infty} A_k$ and each A_k is denumerable. Therefore, $\mathbb{Q}^+$ is countable.

Similarly, $\mathbb{Q}^-$ is also countable, since $\exists$ a bijection $f : \mathbb{Q}^+ \to \mathbb{Q}^-$ defined by
$f(x) = -x,\ \forall\ x \in \mathbb{Q}^+$.

Hence, $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ is countable. $\square$

Remark 1.8. *All rational numbers in $(0, 1)$ is countable.*

Theorem 1.8. ***The set $\mathbb{R}$ of all real numbers is uncountable.***

Proof. Let $I := (0, 1) \subset \mathbb{R}$. If the assertion is false, then the set I is countable. So, $\exists$ a bijective mapping $f : \mathbb{N} \to I$, $\ni I = \{f(1), f(2), f(3), \cdots f(m), \cdots\}$. Since each $f(m)$ ($m \in \mathbb{N}$) is lying between 0 and 1, therefore by decimal expansion, we can expressed the elements of I as

$$
\begin{aligned}
&f(1) = 0.d_{11}d_{12}d_{13}d_{14}\cdots d_{1n}\cdots,\\
&f(2) = 0.d_{21}d_{22}d_{23}d_{24}\cdots d_{2n}\cdots,\\
&f(3) = 0.d_{31}d_{32}d_{33}d_{34}\cdots d_{3n}\cdots,\\
&\cdots,\quad \cdots,\quad \cdots,\quad \cdots,\quad \cdots,\\
&\cdots,\quad \cdots,\quad \cdots,\quad \cdots,\quad \cdots,\\
&f(m) = 0.d_{m1}d_{m2}d_{m3}d_{m4}\cdots d_{mn}\cdots,\\
&\cdots,\quad \cdots,\quad \cdots,\quad \cdots,\quad \cdots,\\
&\cdots,\quad \cdots,\quad \cdots,\quad \cdots,\quad \cdots,
\end{aligned}
$$

where $0 \le d_{ij} \le 9$ for every $i, j \in \mathbb{N}$.

Now consider a real number $a = 0.a_1a_2a_3a_4\cdots a_n\cdots$, where $0 < a_n < 9$ but $a_n \neq d_{nn}$ for every $n \in \mathbb{N}$.

Then for any $n \in \mathbb{N}$,

$a \neq f(n)$ since a_1 and $f(n)$ differ in n-th decimal place.

Thus we get a real number $a \in (0,1)$ which does not belong I.

Thus the set I is uncountable. Hence the set $\mathbb{R}$ is uncountable. □

Theorem 1.9. ***The set of all irrational numbers is uncountable.***

Proof. Now, $\mathbb{Q}^c := \mathbb{R} - \mathbb{Q}$, the set of all irrational numbers and $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$.

If we assume that the assertion is false, then the set $\mathbb{Q}^c$ of all irrational numbers will be countable. But the set $\mathbb{Q}$ of all rational numbers is countable and so, their union $\mathbb{Q} \cup \mathbb{Q}^c = \mathbb{R}$ of all real numbers will be countable, which is a contradiction. Hence, our assumption is false and the set of all irrational numbers is uncountable. □

Theorem 1.10. ***Let $f : X \to Y$ be onto function. If X is countable, then Y is also countable .***

Proof. Since f is onto, for every $b \in Y$, $\exists$ at least one pre-image $a \in X$ such that $f(a) = b$. Writing one such a as c, we can construct a mapping $g : Y \to X$ by $g(b) = c$. Then the range of g is a subset of the countable set X, and g is one-to-one. So domain of g namely, the set Y must be countable. □

Theorem 1.11. ***Let $f : X \to Y$ be onto function. If the set $f(X)$ is uncountable, then the set X is also uncountable .***

Proof. If we assume that the assertion is false, then the set X will be countable, whereas the range $f(X)$ is uncountable. Now, the function $f : X \to f(X)$ be an onto function. If X is countable, then $f(X)$ will be countable, which is a contradicts the given hypothesis that the set $f(X)$ is uncountable. Hence, our assumption is false and the set X is uncountable. □

Remark 1.9. *The set $\mathbb{R}$ is a superset of the set $\mathbb{Q}$. The set $\mathbb{Q}$ is countable whereas the set $\mathbb{R}$ is uncountable. So, every superset of a countable set is not necessarily a countable set.*

Problems 1.1.

Show that

1. Supremum, or *l.u.b.* of a nonempty set of real numbers, whenever it exits, is unique.
2. Infimum, or *g.l.b.* of a nonempty set of real numbers, whenever it exits, is unique.
3. The upper bound M of a non-empty set S in $\mathbb{R}$ is the supremum of S, iff for every $\epsilon > 0$, $\exists$ a $v \in S$, $\ni$ $M - \epsilon < v \leq M$.
4. The lower bound m of a non-empty set S in $\mathbb{R}$ is the infimum of S, iff for every $\epsilon > 0$, $\exists$ a $v \in S$, $\ni$ $m \leq v < m + \epsilon$.
5. The set $\mathbb{N}$ is not bounded above.
6. $\inf S \leq \inf T \leq \sup T \leq \sup S$, where $T \neq \phi$ such that $T \subseteq S \subseteq \mathbb{R}$ and S is bounded.
7. The set $T = \{-x : x \in S\}$ is bounded below and $\inf T = -\sup S$, where $S \subseteq \mathbb{R}$ is non-empty and bounded above.
8. The set $T = \{-x : x \in S\}$ is bounded above and $\sup T = -\inf S$, where $S \subseteq \mathbb{R}$ is non-empty and bounded below.
9. The set $T = \{x - y : x, y \in S\}$ is bounded and $\sup T = \sup S - \inf S$; $\inf T = \inf S - \sup S$, where $S \subseteq \mathbb{R}$ is non-empty and bounded.
10. The set $T = \{|x - y| : x, y \in S\}$ is bounded above and $\sup T = \sup S - \inf S$, where $S \subseteq \mathbb{R}$ is non-empty and bounded.
11. The set $T = \{x+y : x \in S_1, y \in S_2\}$ is bounded and $\sup T = \sup S_1 + \sup S_2$, $\inf T = \inf S_1 + \inf S_2$, where $S_1, S_2 \subseteq \mathbb{R}$ are non-empty and bounded.
12. The set $T = \{x.y : x \in S_1, y \in S_2\}$ is bounded and $\sup T = \sup S_1 . \sup S_2$, $\inf T = \inf S_1 . \inf S_2$, where $S_1, S_2 \subseteq \mathbb{R}^+$ are non-empty and bounded.
13. $\sup S_1 \cup S_2 = \max\{\sup S_1, \sup S_2\}$ and $\inf S_1 \cup S_2 = \min\{\inf S_1, \inf S_2\}$, where $S_1, S_2 \subseteq \mathbb{R}$ are non-empty and bounded.
14. $S_1 \cup S_2$ and $S_1 \cap S_2$ are bounded, when $S_1, S_2 \subseteq \mathbb{R}$ are bounded.
15. $x = 0$, if $0 \leq x < \dfrac{1}{n}$ for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$.
16. $x \leq y$, if $x \leq y + \dfrac{1}{n}$ for every $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$.
17. $x = y$, if $0 \leq x - y < \epsilon$ for every $\epsilon > 0$ and $x, y \in \mathbb{R}$.

18. $x = y$, if $|x - y| < \epsilon$ for every $\epsilon > 0$ and $x, y \in \mathbb{R}$.
19. $\sup S \leq \sup T$, where and $S, T \subseteq \mathbb{R}$ are non-empty such that $x \leq y, \forall x \in S,\ y \in T$.
20. $\sup\{x \in \mathbb{Q} : x < a\} = a$, for each $a \in \mathbb{R}$.
21. $\sup\{cx : x \in S\} = c \sup S$ and $\inf\{cx : x \in S\} = c \inf S$, if $c > 0$, where $S \subseteq \mathbb{R}$ is non-empty and bounded.
22. $\sup\{cx : x \in S\} = c \inf S$ and $\inf\{cx : x \in S\} = c \sup S$, if $c < 0$, where $S \subseteq \mathbb{R}$ is non-empty and bounded.
23. $\inf\{1/n : n \in \mathbb{N}\} = 0$ and hence show that $\exists\ n \in \mathbb{N}$ such that $0 < \dfrac{1}{n} < x$ for every $x > 0$.
24. There exits a natural number $n \in \mathbb{N}\}$ such that $0 < \dfrac{1}{2^n} < x$ for every $x > 0$.
25. $|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2),\ \forall x, y \in \mathbb{R}$.
26. $|x + y| = |x| + |y|$, if and only if $xy > 0\ \forall x, y \in \mathbb{R}$.
27. $|x + y| < |x| + |y|$, if and only if $xy < 0\ \forall x, y \in \mathbb{R}$.
28. $\dfrac{|x + y|}{1 + |x + y|} \leq= \dfrac{|x|}{1 + |x|} + \dfrac{|y|}{1 + |y|},\ \forall x, y \in \mathbb{R}$.
29. $\dfrac{|x + y|}{2 + |x| + |y|} \leq= \dfrac{|x|}{1 + |x|} + \dfrac{|y|}{1 + |y|},\ \forall x, y \in \mathbb{R}$.
30. $\max\{x, y\} = \frac{1}{2}(x + y + |x + y|)$ and $\min\{x, y\} = \frac{1}{2}(x + y - |x - y|),\ \forall x, y \in \mathbb{R}$.

****