Lecture Note: Point Set Topology on $\mathbb{R}$

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1.1 Intervals

In view of the ordered field structure of $\mathbb{R}$, types of special subset of $\mathbb{R}$ are called intervals. If $a, b \in \mathbb{R}$ satisfy $a < b$, then

(i) the **open interval** is the set $(a, b) := \{x \in \mathbb{R} : a < x < b\}$.

(ii) the **closed interval** is the set $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$.

(iii) the **half-open interval** or **half-closed** interval is each of the sets $(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$, $\quad [a, b) := \{x \in \mathbb{R} : a \leq x < b\}$.

Here, the points a, b are called the **end points** of the interval. Each of the above four intervals is bounded and has **length** defined by $b - a$. In particular, if $a = b$, the corresponding open interval is the empty set $(a, a) = \phi$, and the corresponding closed interval is the singleton set $[a, a] = \{a\}$.

The unbounded intervals can be defined after replacing a end point or both end points of the intervals by the symbols ∞ ($or + \infty$) and $-\infty$ as a notational convenience. If $a, b \in \mathbb{R}$, then

(i) the **infinite open intervals** are the sets $(a, \infty) := \{x \in \mathbb{R} : x > a\}$, and $(-\infty, b) := \{x \in \mathbb{R} : x < b\}$,

(ii) the **infinite closed intervals** are the sets $[a, \infty) := \{x \in \mathbb{R} : x \geq a\}$, and $(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}$,

(ii) the **infinite** (both closed and open) **interval** is the entire set $\mathbb{R}$ i. e., $(-\infty, \infty) := \mathbb{R}$.

1.2 Neighbourhood of a point

Let $c \in \mathbb{R}$, and $\delta > 0$. Then the set of all points in the open interval $(c - \delta, c + \delta)$ is called δ-neighbourhood of the point c, which is denoted by $N_\delta(c)$. The open interval $N_\epsilon(c) := (c - \epsilon, c + \epsilon)$ is called ϵ-neighbourhood of the point c for every $\epsilon > 0$.

Example : 0.001-neighbourhood of a point 5 is the open interval $(5 - 0.001, 5 + 0.001)$.

In general, a set $S \subseteq \mathbb{R}$ is said to be a neighbourhood of a point $c \in S$, if $\exists$ an open interval (a, b), $\ni$ $c \in (a, b) \subseteq S$. such a neighbourhood is denoted by simply $N(c)$. Thus any open interval containing a point $c \in \mathbb{R}$ must be a neighbourhood of the point c. In other words, a set S is a neighbourhood of a point c, if and only if $\exists$ a natural number n such that $(c - \frac{1}{n}, c + \frac{1}{n}) \subseteq S$. If S is a neighbourhood of a point c and if $\exists$ a set T, satisfying $S \subseteq T \subseteq \mathbb{R}$, then T is also a neighbourhood of a point c.

Example : We can say that $(1,\ 2)$ is a neighbourhood of the point 1.3, or of the point 1.5, or of the point 1.99 etc.

A closed bounded interval containing a point c may not be a neighbourhood of c. For example, $1 \in [1, 2]$ but $[1, 2]$ is not a neighbourhood of 1.

Conclusions 1.1.

1. The union of two neighbourhoods of a point is also a neighbourhood of the point. [Hints : If S_1 and S_2 be two neighbourhoods of a point $c \in \mathbb{R}$, then $\exists$ open intervals (a_1, b_1), and (a_2, b_2) : $c \in (a_1, b_1) \subseteq S_1$, and $c \in (a_2, b_2) \subseteq S_2$. These two together $\Rightarrow c \in (a_1, b_1) \cup (a_2, b_2) \subseteq S_1 \cup S_2 \Rightarrow c \in (a, b) \subseteq S_1 \cup S_2$, where $a = \min\{a_1, a_2\}$, and $b = \max\{b_1, b_2\} \Rightarrow$ the set $S_1 \cup S_2$ is a neighbourhood of of the point c.]
2. The intersection of two neighbourhoods of a point is also a neighbourhood of the point.
   [Hints : If S_1 and S_2 be two neighbourhoods of a point $c \in \mathbb{R}$ then $\exists$ open intervals (a_1, b_1), and (a_2, b_2) : $c \in (a_1, b_1) \subseteq S_1$, and $c \in (a_2, b_2) \subseteq S_2$. These two together $\Rightarrow c \in (a_1, b_1) \cap (a_2, b_2) \subseteq S_1 \cap S_2 \Rightarrow c \in (a, b) \subseteq S_1 \cap S_2$, where $a = \max\{a_1, a_2\}$, and $b = \min\{b_1, b_2\} \Rightarrow$ the set $S_1 \cap S_2$ is a neighbourhood of of the point c.]
3. The union of finite number of neighbourhoods of a point is also a neighbourhood of the point.
   [Hints : Followed from conclusion 1]
4. The intersection of finite number neighbourhoods of a point is also a neighbourhood of the point.
   [Hints : Followed from conclusion 2]
5. The union of infinite number of neighbourhoods of a point is also a neighbourhood of the point.
   [Hints : Let S_α, where $\alpha \in \Lambda$, the index set, are the neighbourhoods of a point $c \in \mathbb{R}$. If $S = \bigcup_{\alpha \in \Lambda} S_\alpha$, then $S_\alpha \subseteq S$. Now, each S_α is a neighbourhood of $c \Rightarrow S$ is also a neighbourhood of c.]
6. The intersection of infinite number neighbourhoods of a point may not be a neighbourhood of the point.
   [Hints : For Example, $\left(-\dfrac{1}{n}, \dfrac{1}{n}\right)$ is a neighbourhood of a point 0 for every $n \in \mathbb{N}$. But $\bigcap_{n=1}^{\infty} \left(-\dfrac{1}{n}, \dfrac{1}{n}\right) = \{0\}$, which is not a neighbourhood of 0.]

1.3 Interior Points, Exterior points, Boundary Points

Let $S \subseteq \mathbb{R}$. A point $x \in S$ is called an **interior point** of S, if $\exists$ a neighbourhood $N(x)$ of x, $\ni\ N(x) \subset S$. Thus if x is an interior point of S, if $\exists$ a neighbourhood $N(x)$ of x, $\ni\ N(x) \cap S = N(x)$.

The set of all interior points of S is called **interior** of S and it is denoted by int S or, S^i or, S^o, and int $S \subseteq S$ always.

Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called an **exterior point** of S, if $x \in \text{int}S^c$, where $S^c = \mathbb{R} - S$, the complement of S in $\mathbb{R}$. That means, $\exists$ a neighbourhood $N(x)$ of x $\ni\ N(x) \subset S^c$. Thus if x is an exterior point of S, if $\exists$ a neighbourhood $N(x)$ of x $\ni$ $N(x) \cap S = \phi$. An exterior point of a set does not belong to the set.

The set of all exterior points of S is called **exterior** of S and it is denoted by ext S or, S^e, and ext $S \subseteq S^c$ always.

Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called an **boundary point** of S, if it is neither an interior point nor an exterior point of S. That means, if x is a boundary point of S, then for every neighbourhood $N(x)$ of x, $N(x) \cap S \neq \phi$, and $N(x) \cap S^c \neq \phi$ hold together. A boundary point of a set may or may not belong to the set.

The set of all boundary points of S is called **boundary** of S and it is denoted by ∂S or, S^b.

For every set S In respect to $\mathbb{R}$, a point x must be either in int S, or in ext S, or in ∂S. The sets int S, ext S and ∂S are pairwise disjoint in the sense

$$\text{int}S \cap \text{ext}S = \text{ext}S \cap \partial S = \partial S \cap \text{int}S = \phi, \quad \text{and} \quad \text{int}S \cup \text{ext}S \cup \partial S = \mathbb{R}.$$

Examples 1.1.

1. Let $S = \mathbb{N}$, the set of natural numbers. Then no point of S is an interior point of S. Thus, int S =int $\mathbb{N} = \phi$. Again, all points of S^c are the interior points of S^c. Thus, ext S =ext $\mathbb{N} = N^c$. Hence, all points of S are boundary points. Thus, $\partial S = \partial\mathbb{N} = \mathbb{N}$.
2. Let $S = \mathbb{Z}$, the set of integers. Then no point of S is an interior point of S. Thus, int S =int $\mathbb{Z} = \phi$. Again, all points of S^c are the interior points of S^c. Thus, Ext S =ext $\mathbb{Z} = N^c$. Hence, all points of S are boundary points. Thus, $\partial S = \partial\mathbb{Z} = \mathbb{Z}$.
3. Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then no point of S is an interior point of S. Thus, int $S = \phi$. Again, all points of $S^c - \{0\}$ are the interior points of S^c. Thus, Ext $S = S^c - \{0\}$. Hence, all points of $S \cup \{0\}$ are boundary points. Thus, $\partial S = S \cup \{0\}$.
4. Let $S = \mathbb{Q}$, the set of rational numbers. Every neighbourhood of a rational number contains rational as well as irrational numbers. Then no point of S is an interior point of S. Thus, int $S = \phi$. Again, every neighbourhood of an irrational number contains rational as well as irrational numbers. Then no point of S^c is an interior

point of S. Thus, ext $S = \phi$. Hence, all rational and irrational points are the boundary points. Thus, $\partial S = \mathbb{R}$.

5. Let $S = (1, 2)$. Then every point of S is an interior point of S. Thus, int $S = (1, 2)$. Again, all points of $S^c - \{1, 2\}$ are the interior points of S^c. Thus, ext $S = S^c - \{1, 2\} = (-\infty, a) \cup (2, \infty)$. Hence, the points $1, 2$ are the boundary points. Thus, $\partial S = \{1, 2\}$.
6. Let $S = [1, 2]$. Then $N(1) \nsubseteq S$ and $N(2) \nsubseteq S$. Thus every point of S is an interior point of S except 1 and 2. Thus, int $S = (1, 2)$. Again, all points of S^c are the interior points of S^c. Thus, ext $S = S^c$. Hence, the points $1, 2$ are the boundary points. Thus, $\partial S = \{1, 2\}$.
7. Let $S = (-\infty, a]$ or $S = [a, \infty) \;\; ; a \in \mathbb{R}$. Then $N(a) \nsubseteq S$. Thus every point of S is an interior point of S except a. Thus, int $S = S - \{a\}$. Again, all points of S^c are the interior points of S^c. Thus, ext $S = S^c$. Hence, the only point a is the boundary point. Thus, $\partial S = \{a\}$.
8. Let $S = \mathbb{R}$, the set of real numbers. Then every point of S is an interior point of S. Thus, int $S = \mathbb{R}$. Again, $S^c = \phi$, so no point of S^c is a interior point of ϕ. Thus, ext $S = \phi$. Hence, no point of S is a boundary point. Thus, $\partial S = \partial \mathbb{R} = \phi$.
9. Let $S = \phi$, the empty set. Then S has no interior point. Thus, int $S = \phi$. Again, $S^c = \mathbb{R}$, so all the points of S^c are the interior points of S^c. Thus, ext $S = \mathbb{R}$. Hence, no point of S is a boundary point. Thus, $\partial S = \partial \mathbb{R} = \phi$.
10. Let $S = \{a_1, \ a_2, \ \cdots, \ a_n\}$, a finite set, $a_i \in \mathbb{R}, \ \forall \ i = 1, 2, 3, \cdots, n$. Then S has no interior point. Thus, int $S = \phi$. Again, all points of S^c are the interior points of S^c. Thus, ext $S = S^c$. Hence, the all the points in S are the boundary points. Thus, $\partial S = \{a_1, \ a_2, \ \cdots, \ a_n\}$.

Theorem 1.1. ***If*** $A, B \subseteq \mathbb{R}$ ***with*** $A \subseteq B$, ***then*** $\operatorname{int} A \subseteq \operatorname{int} B$.

Proof. If int $A = \phi$, then the theorem is proved trivially.

If int $A \neq \phi$, then let $x \in \operatorname{int} A \Rightarrow x$ is an interior point of A, So, $\exists$ a neighbourhood $N(x)$ of $x, \ni N(x) \subset A \subseteq B \Rightarrow x$ is an interior point of $B \Rightarrow x \in \operatorname{int} B \Rightarrow \operatorname{int} A \subseteq \operatorname{int} B$. □

Theorem 1.2. ***If*** $A, B \subseteq \mathbb{R}$ ***then*** $\operatorname{int}(A \cap B) = \operatorname{int} A \cap \operatorname{int} B$.

Proof. Since, $A \cap B \subseteq A$ and $A \cap B \subseteq B$ also, So, int $(A \cap B) \subseteq$ int A and int $(A \cap B) \subseteq$ int B. These two together $\Rightarrow$ int $(A \cap B) \subseteq$ int $A \cap$ int B.

Again, if int $A \cap$ int $B = \phi$, then int $A \cap$ int $B \subseteq$ int $(A \cap B)$.
If int $A \cap$ int $B \neq \phi$, then let $x \in$ int $A \cap$ int $B \Rightarrow x \in$ int A and $x \in$ int $B \Rightarrow \exists$ neighbourhoods $N_{\delta_1}(x), N_{\delta_2}(x), \ni \ N_{\delta_1}(x) \subset A$ and $N_{\delta_2}(x) \subset B$. If we take $\delta = \min\{\delta_1, \delta_2\}$, then $N_\delta(x) \subset A$ and $N_\delta(x) \subset B \Rightarrow N_\delta(x) \subset A \cap B \Rightarrow x \in$ int $(A \cap B) \Rightarrow$ int $A \cap$ int $B \subseteq$ int $(A \cap B)$.

Hence, int $A \cap$ int $B =$ int $(A \cap B)$. □

Theorem 1.3. ***If*** $A, B \subseteq \mathbb{R}$ ***then*** $\text{int}\, A \cup \text{int}\, B \subseteq \text{int}\,(A \cup B)$.

Proof. Since, $A \subseteq A \cup B$ and $B \subseteq A \cup B$ also, So, int $A \subseteq$ int $(A \cup B)$ and int $B \subseteq$ int $(A \cup B)$. These two together $\Rightarrow$ int $A \cup$ int $B \subseteq$ int $(A \cup B)$. □

1.4 Open Set

A set is $S \subseteq \mathbb{R}$ is said to be an open set, if all its points are interior points. That means, $\forall\, x \in S$, $\exists$ a neighbourhood $N(x)$ of x, $\ni\; N(x) \subset S$.

Theorem 1.4. ***A set*** $S \subseteq \mathbb{R}$ ***is an open set, if and only if*** $\text{int}\, S = S$.

Proof. For any set S, we have from definition int $S \subseteq S$. Again, the set S is open$\Rightarrow$ all its points are interior points $\Rightarrow S \subseteq$ int S. Hence, if S is open, these two together $\Rightarrow$ int S=S.
Conversely, if int S=$S \Rightarrow$ every point of S is an interior point $\Rightarrow S$ is open. □

Examples 1.2.

1. Let $S = \mathbb{N}$. Then no point of S is an interior point of S.
   So, int $S = \phi \neq S$ and hence S is not an open set.
2. Let $S = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \cdots\right\}$. Then no point of S is an interior point of S.
   So, S is not an open set.
3. Let $S = \mathbb{Q}$. Then no point of S is an interior point of S and int $S = \phi \neq S$.
   So, S is not an open set.
4. $S = (1, 2)$. Then every point of S is an interior point of S.
   So, int $S = S$ and S is an open set.
5. $S = [1, 2]$. Though 1, $2 \in S$ but they are not interior points of S.
   So, int $S = \{x \in \mathbb{R} : 1 < x < 2\} = (1, 2) \neq S$ and hence S is not an open set.
6. Let $S = \mathbb{R}$. Then every point of S is an interior point of S.
   So, int $S = S$, and S is an open set.
7. Let $S = \phi$, the empty set. Then S contains no point and it has no interior point.
   So, int $S = \phi = S$, and S is an open set.
8. Let $S = \{a_1,\ a_2,\ \cdots,\ a_n\}$, a finite set, $a_i \in \mathbb{R}$, $\forall\, i = 1, 2, 3, \cdots, n$. Then S has no interior point.
   So, int $S = \phi \neq S$ and hence S is not an open set.

Theorem 1.5. ***An open interval is an open set.***

Proof. Case I : Let $I := (a, b)$ be an open interval, $a, b \in \mathbb{R}$ with $b > a$.
Now, for every $x \in I$, $\Big((a+x)/2, (b+x)/2\Big)$ is a neighbourhood of x. So, writing this nbd as $N(x)$, that means, $N(x) := \Big((a+x)/2, (b+x)/2\Big) \Rightarrow N(x) \subset I \Rightarrow x$ is an interior point of I. Since $x \in I$ is arbitrary, we have I is an open set.

Case II : Let $I := (a, \infty)$ be an open interval, $a \in \mathbb{R}$.
Now, for every $x \in I$, $(a, x+1)$ is a neighbourhood of x. So, writing this nbd as $N(x)$, that means, $N(x) := (a, x+1) \Rightarrow N(x) \subset I \Rightarrow x$ is an interior point of I. Since $x \in I$ is arbitrary, we have I is an open set.

Case III : Let $I := (-\infty, a)$ be an open interval, $a \in \mathbb{R}$.
Now, for every $x \in I$, $(x-1, a)$ is a neighbourhood of x. So, writing this nbd as $N(x)$, that means, $N(x) := (x-1, a) \Rightarrow N(x) \subset I \Rightarrow x$ is an interior point of I. Since $x \in I$ is arbitrary, we have I is an open set.

Case IV : Let $I := (-\infty, \infty)$ be an open interval.
Now, for every $x \in I$, $(x-1, x+1)$ is a neighbourhood of x. So, writing this nbd as $N(x)$, that means, $N(x) := (x-1, x+1) \Rightarrow N(x) \subset I \Rightarrow x$ is an interior point of I. Since $x \in I$ is arbitrary, we have I is an open set. □

Theorem 1.6. ***For every set $S \subseteq \mathbb{R}$, $\operatorname{int} S$ is an open set.***

Proof. If int $S = \phi$, then since ϕ is an open set, int S is an open set.

If int $S = \phi$, then let $x \in$ int $S \Rightarrow x$ is an interior point of $S \Rightarrow \exists$ a neighbourhood $N(x)$ of x, $\ni$ $N(x) \subset S$. Since $N(x)$ is an open interval, every point of $N(x)$ is an interior point of it $\Rightarrow N(x) \subset$ int $S \Rightarrow x$ is an interior point of int S. Since $x \in$ int S is arbitrary, every point of int S is an interior point of it $\Rightarrow$ int S is an open set. □

Theorem 1.7. ***For every set $S \subseteq \mathbb{R}$, $\operatorname{int} S$ is the largest open set contained in S, or, $\operatorname{int} S$ contains every open subset of S.***

Proof. In the last theorem, we have proved that int S is an open set. The statement 'int S is the largest open set contained in S' means, among all open sets $G \subseteq S$, largest of $\{G\} =$ int $S, \Rightarrow \forall\ G \subseteq$ int S.

Let $G \subseteq S$ be any arbitrary. If $G = \phi$, then $G \subset \operatorname{int} S$ and the theorem is proved trivially.

Next, let $G \neq \phi$, and let $x \in G$. Since G is open $\Rightarrow x \in$ int $G \subseteq$ int $S \Rightarrow G \subseteq$ int $S \Rightarrow$ $\operatorname{int} S$ is the largest open set contained in S. □

Theorem 1.8. ***The union of two open sets in $\mathbb{R}$ is also an open set.***

Proof. Let G_1 and G_2 be two open sets in $\mathbb{R}$.

If $G_1 \cup G_2 = \phi$, and since ϕ is an open set, therefore $G_1 \cap G_2$ is an open set.

If $G_1 \cup G_2 \neq \phi$ and then let $x \in G_1 \cup G_2 \Rightarrow x \in G_1$, or $x \in G_2$.

Let $x \in G_1$. Since G_1 is open, x is an interior point of G_1. Therefore, $\exists$ a neighbourhood $N(x)$ of $x \ni N(x) \subset G_1$. Again $G_1 \subseteq G_1 \cup G_2$. So $N(x) \subset G_1 \cup G_2$. $\Rightarrow x$ is an interior point of $G_1 \cup G_2$. Since x is arbitrary, every point of $G_1 \cup G_2$ is an interior point of $G_1 \cup G_2 \Rightarrow G_1 \cup G_2$ is an open set.

Similarly, if $x \in G_2$, we can prove that $G_1 \cup G_2$ is an open set. □

Theorem 1.9. ***The intersection of two open sets in $\mathbb{R}$ is also an open set.***

Proof. Let G_1 and G_2 be two open sets in $\mathbb{R}$.

If $G_1 \cap G_2 = \phi$, and since ϕ is an open set, therefore $G_1 \cap G_2$ is an open set.

If $G_1 \cap G_2 \neq \phi$ and then let $x \in G_1 \cap G_2$. Then $x \in G_1$, and $x \in G_2$. Since G_1 is an open set, x is an interior point of G_1. Hence, $\exists$ a $\delta_1 > 0 \ni N_{\delta_1}(x) \subset G_1$.

Again, since G_2 is an open set, x is an interior point of $G_2 \Rightarrow \exists$ a $\delta_2 > 0 \ni N_{\delta_2}(x) \subset G_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then obviously, $\delta > 0$, and $N_\delta(x) \subseteq N_{\delta_1}(x) \subset G_1$, and $N_\delta(x) \subseteq N_{\delta_2}(x) \subset G_2$. Consequently, $N_\delta(x) \subset G_1 \cap G_2 \Rightarrow x$ is an interior point of $G_1 \cap G_2$. Since x is arbitrary, therefore every point of $G_1 \cap G_2$ is an interior point of $G_1 \cap G_2$. Therefore $G_1 \cap G_2$ is an open set. □

Conclusions 1.2.

1. The union of finite number of open sets is also an open set.

   [Hints : By the above theorem, for every $x \in \bigcup_{i=1}^{n} G_i$, $\exists$ a neighbourhood $N(x), \ni N(x) \subset \bigcup_{i=1}^{n} G_i \Rightarrow \bigcup_{i=1}^{n} G_i$ is an open set.]

2. The intersection of finite number open sets is also an open set.

   [Hints : By the above theorem, for every $x \in \bigcap_{i=1}^{n} G_i$, $\exists$ a neighbourhood $N_\delta(x)$, where $\delta = \min\limits_{1 \leq i \leq n} \{\delta_i\}, \ni N_\delta(x) \subset \bigcap_{i=1}^{n} G_i \Rightarrow \bigcap_{i=1}^{n} G_i$ is an open set.]

3. The union of infinite number of open sets is also an open set.

   [Hints : By the above theorem, for every $x \in \bigcup_{\alpha \in \Lambda} G_\alpha, \Lambda$ being the index set. $\exists$ a neighbourhood $N(x), \ni N(x) \subset \bigcup_{\alpha \in \Lambda} G_i \Rightarrow \bigcup_{\alpha \in \Lambda} G_\alpha$ is an open set.]

4. The intersection of infinite number open sets is not necessarily an open set.

   [Hints : For example, if we consider $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ which is an open set for every $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$, which is not an open set.

Again, if we consider $B_n = (-n, n)$ which is also an open set for every $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} (-n, n) = B_1 = (-1, 1)$, which is an open set.]

5. Every non-empty open set in $\mathbb{R}$ is the union of open intervals.
[Hints : We can write a non-empty open set S as $S = \bigcup_{\alpha \in \Lambda} x_\alpha$, the union of singleton elements x_α, Λ being the index set. Since S is open, $\exists$ a neighbourhood $N_\delta(x_\alpha)$ of x_α, $\ni$ $N_\delta(x_\alpha) \subset S$. Hence, $S = \bigcup_{\alpha \in \Lambda} x_\alpha \subset \bigcup_{\alpha \in \Lambda} N_\delta(x_\alpha) \subset S \Rightarrow S = \bigcup_{\alpha \in \Lambda} N_\delta(x_\alpha)$. As each $N_\delta(x_\alpha)$ is an open interval, hence the conclusion.]

Problems 1.1.

Show that S is an open set in each of the followings.

1. $S = \{x \in \mathbb{R} : x^2 - 5x + 6 < 0\}$.
2. $S = \{x \in \mathbb{R} : x^2 - 5x + 6 > 0\}$.
3. $S = \{x \in \mathbb{R} : \sin x \neq 0\}$.
4. $S = A - B$, where $A = (0, 1]$ and $B = \{1/n, n \in \mathbb{N}\}$.

1.5 Deleted neighbourhood of point

Let $c \in \mathbb{R}$, and $\delta > 0$. Then the set of all points in $(c - \delta, c + \delta) - \{c\}$ is called deleted δ-neighbourhood of the point c, which is denoted by $N'_\delta(c)$. The set $N'_\epsilon(c) := (c - \epsilon, c + \epsilon) - \{c\}$ is called deleted ϵ-neighbourhood of the point c for every $\epsilon > 0$. The set $N(c) - \{c\}$ is called deleted neighbourhood of c, which is denoted by $N'(c)$.

Example : 0.001 deleted-neighbourhood of a point 5 is the set of points in $(5 - 0.001, 5 + 0.001) - \{5\}$, that means the set $(4.999, 5) \cup (5, 5.001)$.

1.6 Limit Point/Accumulation Point/Cluster Point

Let $S \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is said to be a limit point (or cluster point or accumulation point) of S, if every deleted neighbourhood of c contains at least a point of S.

Thus c will be a limit point of S, if for each deleted neighbourhood $N'(c)$, we have $N'(c) \cap S \neq \phi$. A limit point of S may or may not belong to S.

1.7 Isolated Point

Let $S \subseteq \mathbb{R}$. A point $x \in S$ is said to be an isolated point of S, if x in not a limit point of S.

A point x is an isolated point of $S \Leftrightarrow$ the point x is not a limit point of $S \Leftrightarrow \exists$ a deleted neighbourhood $N'(x) \ni \ N'(x) \cap S = \phi$. Since $x \in S$, we have $N(x) \cap S = \{x\}$.

Examples 1.3.

1. Let $S = \mathbb{N}$. Then every point of S is an isolated point of S. Therefore no point of S is a limit point of S.

2. Let $S = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \cdots\right\}$. Then every point of S is an isolated point of S. For every $\epsilon > 0$, $\exists$ a natural number $m \in \mathbb{N}$, $\ni\ 0 < \frac{1}{m} < \epsilon$. Since $\frac{1}{m} \in S$, we have $N'_\epsilon(0) \cap S \neq \phi$. Therefore, 0 is a limit point of S but $0 \notin S$.

3. Let $S = \mathbb{Q}$. Then no point of S is an isolated point of S. Every point of $\mathbb{R}$ is a limit point of S, since each deleted neighbourhood of a point in $\mathbb{R}$ contains at least a point of $\mathbb{Q}$.

4. $S = (1, 2)$. Then no point of S is an isolated point of S. Every point of S is a limit point of S. The points 1 and 2 are also the limit points of S.

5. $S = [1, 2]$. Then no point of S is an isolated point of S. Every point of S is a limit point of S.

6. Let $S = \mathbb{R}$. Then no point of S is an isolated point of S. Every point of S is an limit point of S, since each deleted neighbourhood of a point in $\mathbb{R}$ contains a point of $\mathbb{R}$.

7. Let $S = \phi$, Then no point of S is an isolated point of S and no point is a limit point of S.

8. Let $S = \{0\}$. Then 0 is an isolated point of S and no point is a limit point of S.

Conclusions 1.3.

1. If c be a limit point of $S \subseteq \mathbb{R}$, then every neighbourhood of the point c contains infinitely many points of S.

Proof : c is a limit point of $S \Rightarrow \exists$ a deleted neighbourhood $N'(c)$ of $c :\ N'(c) \cap S \neq \phi$. Let $T = N'(c) \cap S \neq \phi$. Then $T \neq \phi$. We prove that T contains infinitely many points of S.

If not, let $T = \{x_1, x_2, \cdots, x_m\}$, a finite set.

Let $\delta_i = |x_i - c|$ for $i = 0, 1, 2, \cdots, n$. Then each $\delta_i > 0$, since $x_i \neq c$, $(x_i \in T,$ and $c \neq T)$. If we take $\delta = \min_i \{\delta_i\} > 0$, then obviously $x_i \notin N_\delta(c)$ for $i = 0, 1, 2, \cdots, n$. This implies that $N'_\delta(c) \cap S = \phi \Rightarrow c$ is not a limit point of $S \Rightarrow T$ is not finite $\Rightarrow N'_\delta(c)$ contains infinitely many points of S.

2. A finite set has no limit point.

Proof : As neighbourhood of a limit point contains infinitely many points. So a finite set has no limit point.

3. The set $\mathbb{N}$ has no limit point.

Proof : For any $c \in \mathbb{R}$, the neighbourhood $(c-1, c+1)$ of c contains at most two points of $\mathbb{N}$ which is against that every As neighbourhood of a limit point contains infinitely many points. So the set $\mathbb{N}$ has no limit point.

4. If $S \subseteq \mathbb{R}$, then every interior point of S is a limit point of S but the converse is not always true.

Proof : If x is an interior point of S, then $\exists$ a neighbourhood $N(x)$ of x : $N(x) \subset S$. Since $N(x)$ contains infinitely many points of S. So $N'(x) \cap S \neq \phi \Rightarrow x$ is a limit point of S.

For converse, example : The point 0 is a limit point of the set $S = \{1/n : n \in \mathbb{N}\}$ which is not an interior point of the set S.

5. If $S \subseteq \mathbb{R}$ is a bounded and $\inf S, \sup S \notin S$, then both $\inf S$, and $\sup S$ are the limit points of S.

Proof : Let $M = \sup S \Rightarrow x \leq M,\ \forall\ x \in S$, and $\exists$ a $y \in S$: $y > M - \epsilon$ for every $\epsilon > 0$. $\Rightarrow M - \epsilon < y \leq M < M + \epsilon \Rightarrow$ The neighbourhood $(M - \epsilon, M + \epsilon)$ of M contains a point y other than M. Since ϵ is arbitrary, M is a limit point of S.

Theorem 1.10 (**Bolzano-Weierstrass Theorem**). ***Every bounded, infinite subset of $\mathbb{R}$ has a limit point.***

Proof. Let $S \subset \mathbb{R}$ be bounded and infinite set. Since S is bounded, there exist $a, b \in \mathbb{R}$, with $a < b$, $\ni S \subseteq [a, b]$. Let $b - a = l$ and denote the midpoint of the interval $[a, b]$ by m. Since S is infinite, at least one of $[a, m]$ and $[m, b]$ must contain infinitely many points of S. Select an interval satisfying this condition and denoting its left endpoint by a_1 and its right endpoint by b_1. Observe that $[a_1, b_1] \subset [a, b]$ and $b_1 - a_1 = \dfrac{l}{2}$. Denoting the midpoint of the interval $[a_1, b_1]$ by m_1, we can conclude that at least one of $[a_1, m_1]$ and $[m_1, b_1]$ must contain infinitely many points of S. Select an interval satisfying this condition and denoting its left endpoint by a_2 and its right endpoint by b_2. Observe that $[a_2, b_2] \subset [a_1, b_1]$ and $b_2 - a_2 = \dfrac{l}{2^2}$. Continuing this process inductively, for each $n \in \mathbb{N}$, we have an interval $[a_n, b_n]$, which contains infinitely many points of S and $b_n - a_n = \dfrac{l}{2^n}$, satisfying

$[a_n, b_n] \subset [a_{n-1}, b_{n-1}] \subset \cdots \subset [a_1, b_1] \subset [a, b]$,

Next, we define a set $A := \{a_1,\ a_2,\ \cdots\ , a_n,\ \cdots\} \subset [a, b]$. So A is bounded, and as such, has a supremum which we denote by ξ. So, given $\epsilon > 0$, there exists a $k \in \mathbb{N}$, $\ni$ $\xi - \epsilon < a_k \leq \xi$. Now, we select an $n > k$, $\ni \dfrac{l}{2^n} < \epsilon$, Furthermore, we see that $a_n \geq a_k$ for every $n > k$. Thus, we have

$\xi - \epsilon < a_k \leq a_n < b_n = a_n + \dfrac{l}{2^n} < \xi + \epsilon, \quad \forall\ \ n > k.$

Since $[a_n, b_n] \subset (\xi - \epsilon, \xi + \epsilon)$ contains infinitely many points of S, $\forall \ n > k$. Hence, we conclude that ξ is a limit point of S. □

Alternative Proof. Let $S \subset \mathbb{R}$ be bounded and infinite set. Suppose S has no limit point. Let $X \subseteq S$ be non-empty. Since X is bounded, we have $\inf X \in X$, $\sup X \in X$, otherwise, $\inf X, \sup X$ will be the limit points of X, hence also the limit points of S. In particular, we define the set X as $X := \{x_n, \ n \geq 0\}$, where $x_n := \inf\Big\{S - \{x_0, x_1, \cdots, x_{n-1}\}\Big\}$, $\forall \ n \in \mathbb{N}$ with $x_0 := \inf\{S\}$. Now, for every $n \geq 0$, $x_n \in S$ which is not a limit point of S, and $X \subseteq S$ be a non-empty, bounded set. It is obvious that $\{x_n\}$ is a strictly increasing infinite sequence of members of X. So, $\sup X \notin X$ which is a contradiction of above. So the set S has a limit point. □

1.8 Derived Set

The set of all limit points of a set $S \subseteq \mathbb{R}$ is said to be derived set of S and it is denoted by S'.

Examples 1.4.

1. Let $S = \mathbb{N} = \{1, \ 2, \ 3, \ \cdots\}$. No point of S is a limit point of S. Thus, $S' = \phi$.
2. Let $S = \left\{1, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \cdots\right\}$. The point 0 is a limit point of S. Thus, $S' = \{0\}$.
3. Let $S = \mathbb{Q}$. Every point of $\mathbb{R}$ is a limit point of S. Thus, $S' = \mathbb{R}$.
4. $S = \{x \in \mathbb{R} : 1 < x < 2\}$. Set of limit points is the closed interval $[1, 2]$.
Thus, $S' = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$.
5. $S = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$. Set of limit points is the closed interval $[1, 2]$.
Thus, $S' = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$.
6. Let $S = \mathbb{R}$. Every point of S is a limit point of S. Thus, $S' = \mathbb{R}$.
7. Let $S = \phi$, No point is a limit point of S. Thus, $S' = \phi$.
8. Let $S = \{0\}$. No point is a limit point of S. Thus, $S' = \phi$.

1.8.1 Problems :

Find S' in each of the followings.

1. $S = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$. Answer : $S' = \{1\}$.
2. $S = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$. Answer : $S' = \{1\}$.

3. $S = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$. Answer : $S' = \{1\}$.
4. $S = \left\{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\right\}$. Answer : $S' = \{-1, 1\}$.
5. $S = \left\{(-1)^n \left(1 + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$. Answer : $S' = \{-1, 1\}$.
6. $S = \left\{m + \frac{1}{n} : m, n \in \mathbb{N}\right\}$. Answer : $S' = \mathbb{N}$.
7. $S = \left\{m + \frac{1}{5^n} : m, n \in \mathbb{N}\right\}$. Answer : $S' = \mathbb{N}$.
8. $S = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\}$. Answer : $S' = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.
9. $S = \left\{\frac{1}{5^m} + \frac{1}{5^n} : m, n \in \mathbb{N}\right\}$. Answer : $S' = \{0\} \cup \left\{\frac{1}{5^n} : n \in \mathbb{N}\right\}$.
10. $S = \left\{\frac{(-1)^m}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\}$. Answer : $S' = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{\frac{-1}{2n-1} : n \in \mathbb{N}\right\}$.

1.9 Closed Set

A set $S \subseteq \mathbb{R}$ is said to be closed set, if it contains all its limit points.
Thus a set is closed iff $S' \subseteq S$.

Examples 1.5.

1. Let $S = \mathbb{N} = \{1, 2, 3, \cdots\}$. Then $S' = \phi \subseteq S$. Thus, S is closed.
2. Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$. Then $S' = \{0\} \subseteq S$. Thus, S is not closed.
3. Let $S = \mathbb{Q}$. Then $S' = \mathbb{R} \nsubseteq S$. Thus, S is not closed.
4. $S = \{x \in \mathbb{R} : 1 < x < 2\}$. Then $S' = \{x \in \mathbb{R} : 1 \leq x \leq 2\} \nsubseteq S$. Thus, S is not closed.
5. $S = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$. Then $S' = \{x \in \mathbb{R} : 1 \leq x \leq 2\} \subseteq S$. Thus, S is closed.
6. Let $S = \mathbb{R}$. Then $S' = \mathbb{R} \subseteq S$. Thus, S is closed.
7. Let $S = \phi$, Then $S' = \phi \subseteq S$. Thus, S is closed.
8. Let $S = \{0\}$. Then $S' = \phi \subseteq S$. Thus, S is closed.

Remark 1.1. *A closed interval is a closed set*

Theorem 1.11. ***The complement of an open set in $\mathbb{R}$ is a closed set.***

Proof. Let A be an open set in $\mathbb{R}$.

If $A = \phi$, which is an open set. So $A^c = \mathbb{R}$, which is a closed set. Hence the theorem.

Let $A \neq \phi$, and $x \in A$. Then x must be an interior point of A. So, $\exists$ a neighbourhood $N(x)$ of x, $\ni$ $N(x) \subset A$. This implies that $N(x) \nsubseteq A^c \Rightarrow N(x) \cap A^c = \phi \Rightarrow N'(x) \cap A^c = \phi$. This shows that x is not a limit point of $A^c \Rightarrow x \notin (A^c)'$.

Thus if $x \in A \Rightarrow x \notin (A^c)'$. Contrapositively, for any $x \in (A^c)' \Rightarrow x \notin A \Rightarrow x \in A^c \Rightarrow (A^c)' \subseteq A^c \Rightarrow A^c$ is a closed set. □

Theorem 1.12. ***The complement of an closed set in $\mathbb{R}$ is an open set.***

Proof. Let A be a closed set in $\mathbb{R}$.

If $A = \mathbb{R}$, which is a closed set. So $A^c = \phi$, which is an open set. Hence the theorem.

Let $A \neq \mathbb{R}$ so that $A^c \neq \phi$, and let $x \in A^c$. Then $x \notin A$. Since A is closed, x is not a limit point of A. So, $\exists$ a deleted neighbourhood $N'(x)$ of x, $\ni$ $N'(x) \cap A = \phi$. This implies that $N'(x) \nsubseteq A \Rightarrow N'(x) \subseteq A^c \Rightarrow N(x) \subseteq A^c$, since $x \in A^c$. This shows that x is an interior point of A^c. Since $x \in A^c$ is arbitrary, we have A^c is an open set. □

Conclusions 1.4.

1. A set in $\mathbb{R}$ is closed, iff its complement in $\mathbb{R}$ is open.
1. A set in $\mathbb{R}$ is open, iff its complement in $\mathbb{R}$ is closed.

Theorem 1.13. ***The union of two closed sets in $\mathbb{R}$ is also a closed set.***

Proof. Let A, and B be two closed sets in $\mathbb{R}$. Then A^c, and B^c are two open sets in $\mathbb{R}$. So their intersection $A^c \cap B^c$ is also an open set $\Rightarrow (A \cup B)^c$ is an open set $\Rightarrow A \cup B$ is a closed set. □

Theorem 1.14. ***The intersection of two closed sets in $\mathbb{R}$ is also a closed set.***

Proof. Let A, and B be two closed sets in $\mathbb{R}$. Then A^c, and B^c are two open sets in $\mathbb{R}$. So their union $A^c \cup B^c$ is also an open set $\Rightarrow (A \cap B)^c$ is an open set $\Rightarrow A \cap B$ is a closed set. □

Conclusions 1.5.

1. The union of finite number of closed sets is also a closed set.
2. The intersection of finite number closed sets is also a closed set.
3. The intersection of infinite number closed sets is also a closed set.
4. The union of infinite number of closed sets is not necessarily a closed set.

Example : **(i)** $A_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$ is a closed set for every $n \in \mathbb{N}$.

But $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left[-\frac{1}{n},\ \frac{1}{n}\right] = A_1 = [-1, 1]$, which is a closed set.

(ii) $B_n = \left[-1 + \frac{1}{n},\ 1 - \frac{1}{n}\right]$ is a closed set for every $n \in \mathbb{N}$.

But $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \left[-1 + \frac{1}{n},\ 1 - \frac{1}{n}\right] = (-1, 1)$, which is not a closed set.

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